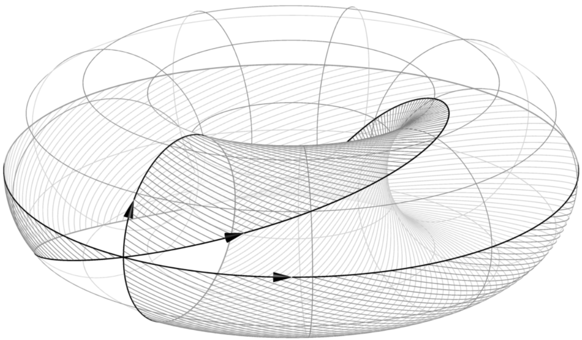
NP-hardness of colouring certain graphs with 3 colours via homotopy

Jakub Opršal



UNIVERSITY OF BIRMINGHAM

Is there a polynomial time algorithm that colours a given 3-colourable graph by 3 colours?

Is there a polynomial time algorithm that colours a given 3-colourable graph by 3 colours?

No! (Unless P = NP) [Karp, 1972]

Is there a polynomial time algorithm that colours a given 3-colourable graph by 4 colours?

No! (Unless P $=$ NP) [Khanna, Linial, Safra, 2000]

Is there a polynomial time algorithm that colours a given 3-colourable graph by 5 colours?

(We didn't know.)

Is there a polynomial time algorithm that colours a given 3-colourable graph by 6 colours?

(We don't know.)

Is there a polynomial time algorithm that colours a given 3-colourable graph by 7 colours?

(We don’t know.)

Is there a polynomial time algorithm that colours a given 3-colourable graph by 1729 colours?

(We don’t know.)

Is there a polynomial time algorithm that colours a given 3-colourable graph by 2^{15} colours?

(We don't know.)

Is there a polynomial time algorithm that colours a given 3-colourable graph by $O(\log n)$ colours?

(We don’t know.)

Is there a polynomial time algorithm that colours a given 3-colourable graph by $O(n^{<1/5})$ colours?

Yes! [Kawarabayashi, Thorup, 2017]

Is there a polynomial time algorithm that colours a given 3-colourable graph by $O(\sqrt{n})$ colours?

Yes! [Wigderson, 1982]

Is there a polynomial time algorithm that colours a given 3-colourable graph by 5 colours?

No! (Unless P $=$ NP) [Bulín, Krokhin, O., 2019]

A black box: Algebraic approach

A black box: Algebraic approach

Theorem [Bulín, Krokhin, O., '19].

Let Γ *and* Δ *be two promise CSPs. If there is a minion homomorphism* $\mathrm{pol}(\Delta) \to \mathrm{pol}(\Gamma)$*, then there is a log-space reduction from* Γ *to* Δ*.*

(pol denotes the minion of all polymorphisms of the problem)

A black box: Algebraic approach

Theorem [Bulín, Krokhin, O., '19].
Let Γ and Δ be two promise CSPs. If there is a minion homomorphism $\text{pol}(\Delta) \to \text{pol}(\Gamma)$, then there is a log-space reduction from Γ to Δ.
(pol denotes the minion of all polymorphisms of the problem)

There is a minion homomorphism $\text{pol}(\Delta) \to \text{pol}(\Gamma)$ where

- Δ is the problem of 27480-colouring 2-colourable 3-uniform hypergraphs,

A black box: Algebraic approach

Theorem [Bulín, Krokhin, O., '19].

Let Γ *and* Δ *be two promise CSPs. If there is a minion homomorphism* $\mathrm{pol}(\Delta) \to \mathrm{pol}(\Gamma)$*, then there is a log-space reduction from* Γ *to* Δ*.*

(pol denotes the minion of all polymorphisms of the problem)

There is a minion homomorphism $\mathrm{pol}(\Delta) \to \mathrm{pol}(\Gamma)$ where

- Δ is the problem of 27480-colouring 2-colourable 3-uniform hypergraphs, which was proven to by NP-hard by [Dinur, Regev, Smyth, '05].
- Γ is 5-colouring 3-colourable graphs.

A black box: Algebraic approach

Theorem [Bulín, Krokhin, O., '19].

Let Γ and Δ be two promise CSPs. If there is a minion homomorphism $\text{pol}(\Delta) \to \text{pol}(\Gamma)$, then there is a log-space reduction from Γ to Δ.

(pol denotes the minion of all polymorphisms of the problem)

There is a minion homomorphism $\text{pol}(\Delta) \to \text{pol}(\Gamma)$ where

- Δ is the problem of 27480-colouring 2-colourable 3-uniform hypergraphs, which was proven to by NP-hard by [Dinur, Regev, Smyth, '05].
- Γ is 5-colouring 3-colourable graphs.

Corollary [Bulín, Krokhin, O., '19].

Colouring graphs that are promised to be 3-colourable with 5 colours is NP-hard. ∎

Chapter I: The story begins. . .

Chapter I: The story begins. . .

If colouring 3-colourable graphs with 3-colours is NP-hard, can we give a stronger promise to make the problem easier?

Andrei Krokhin & O.

Chapter I: The story begins. . .

If colouring 3-colourable graphs with 3-colours is NP-hard, can we give a stronger promise to make the problem easier?

Andrei Krokhin & O.

What stronger promise can we give?

What stronger promise can we give?

Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, a graph homomorphism $G \to H$ is a mapping $h\colon V_G \to V_H$ that preserves edges,

$$(u, v) \in E_G \Rightarrow (h(u), h(v)) \in E_H.$$

What stronger promise can we give?

Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, a graph homomorphism $G \to H$ is a mapping $h\colon V_G \to V_H$ that preserves edges,

$$(u, v) \in E_G \Rightarrow (h(u), h(v)) \in E_H.$$

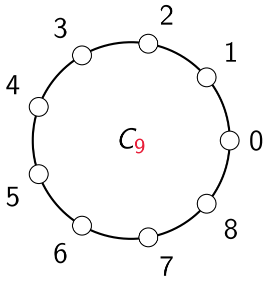
Example. A colouring of a graph G with k colours is just a homomorphism $c\colon G \to K_k$.

What stronger promise can we give?

Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, a graph homomorphism $G \to H$ is a mapping $h\colon V_G \to V_H$ that preserves edges,

$$(u, v) \in E_G \Rightarrow (h(u), h(v)) \in E_H.$$

Example. A colouring of a graph G with k colours is just a homomorphism $c\colon G \to K_k$.



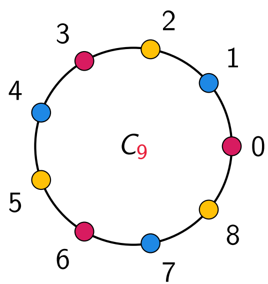
Promise that the input graph maps to an odd cycle C_{2k+1}!

What stronger promise can we give?

Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, a graph homomorphism $G \to H$ is a mapping $h\colon V_G \to V_H$ that preserves edges,

$$(u, v) \in E_G \Rightarrow (h(u), h(v)) \in E_H.$$

Example. A colouring of a graph G with k colours is just a homomorphism $c\colon G \to K_k$.



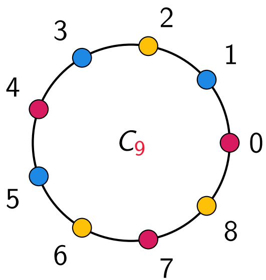
Promise that the input graph maps to an odd cycle C_{2k+1}!

What stronger promise can we give?

Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, a graph homomorphism $G \to H$ is a mapping $h\colon V_G \to V_H$ that preserves edges,

$$(u, v) \in E_G \Rightarrow (h(u), h(v)) \in E_H.$$

Example. A colouring of a graph G with k colours is just a homomorphism $c\colon G \to K_k$.



Promise that the input graph maps to an odd cycle C_{2k+1}!

The problem and its polymorphisms

The problem and its polymorphisms

The problem PCSP(C_{2k+1}, K_3).

Given a graph G that is promised to map to C_{2k+1}, find a 3-colouring:

$$c\colon G \to K_3$$

The problem and its polymorphisms

The problem PCSP(C_{2k+1}, K_3).

Given a graph G that is promised to map to C_{2k+1}, find a 3-colouring:

$$c\colon G \to K_3$$

Definition.

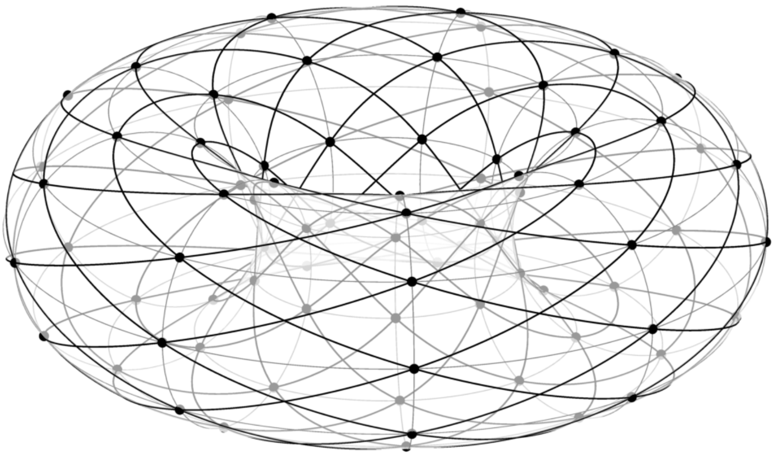
A polymorphism of PCSP(C_{2k+1}, K_3) is a homomorphism $f\colon C_{2k+1}^n \to K_3$, i.e., a mapping $f\colon [2k+1]^n \to \{\bullet, \bullet, \bullet\}$ such that

$$(f(u_1, \dots, u_n), f(v_1, \dots, v_n)) \in E_{K_3}$$

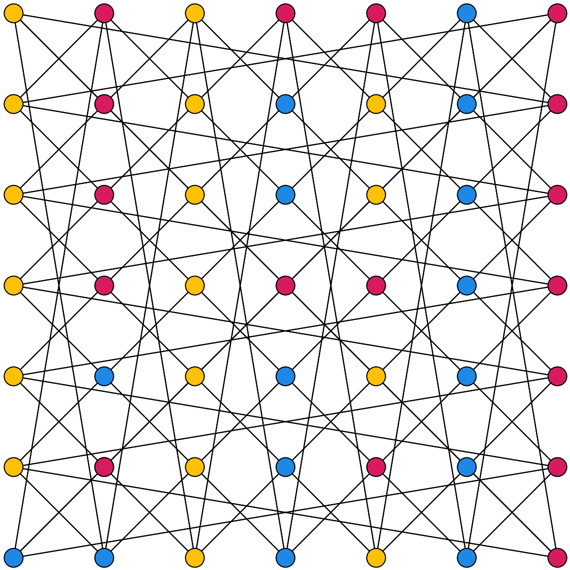
whenever $(u_i, v_i) \in E_{C_{2k+1}}$ for all $i \in [n]$.

$$\mathrm{pol}(C_{2k+1}, K_3) = \{f\colon C_{2k+1}^n \to K_3 \mid n = 1, 2, \dots\}$$

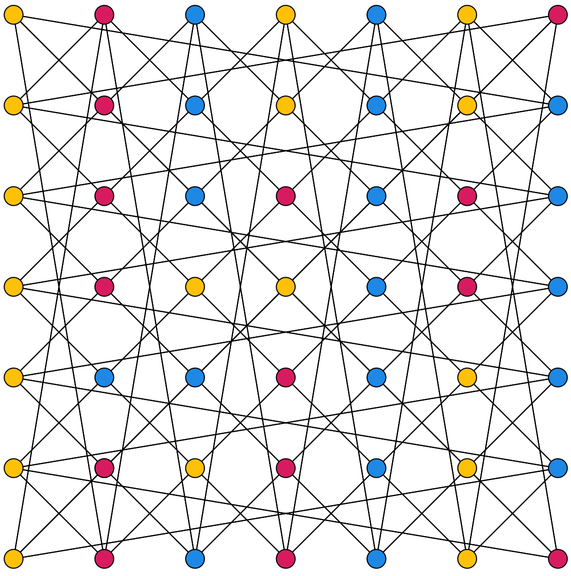
$C_9{}^2$



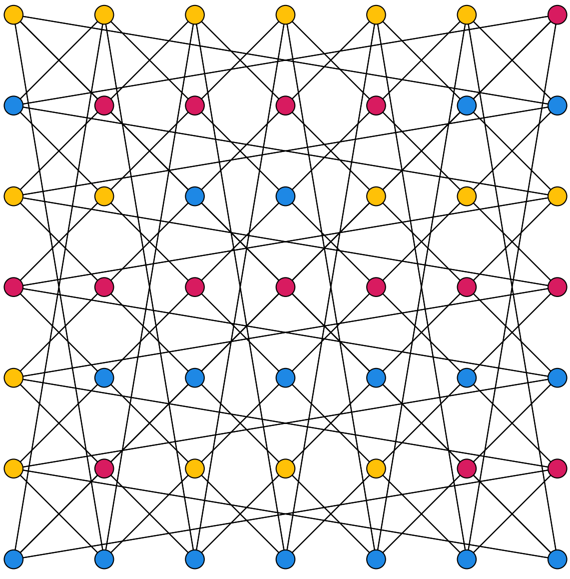
Polymorphisms ${C_7}^2 \to K_3$



Polymorphisms $C_7^{\,2} \to K_3$



Polymorphisms $C_7{}^2 \to K_3$



Two continuous functions $f, g\colon X \to Y$ are said to be

homotopic

if is there is a continuous function $H\colon X \times [0,1] \to Y$ such that $H(0,x) = f(x)$ and $H(1,x) = g(x)$.

From graphs to topological spaces

From graphs to topological spaces

Graph $\longrightarrow$ **Top**

From graphs to topological spaces

Graph $\to$ Top

For a finite set V, Δ^V is the standard simplex with V vertices, i.e.,

$$\Delta^V = \{\lambda \in [0,1]^V : \sum_{v \in V} \lambda_v = 1\}.$$

From graphs to topological spaces

Graph $\to$ Top

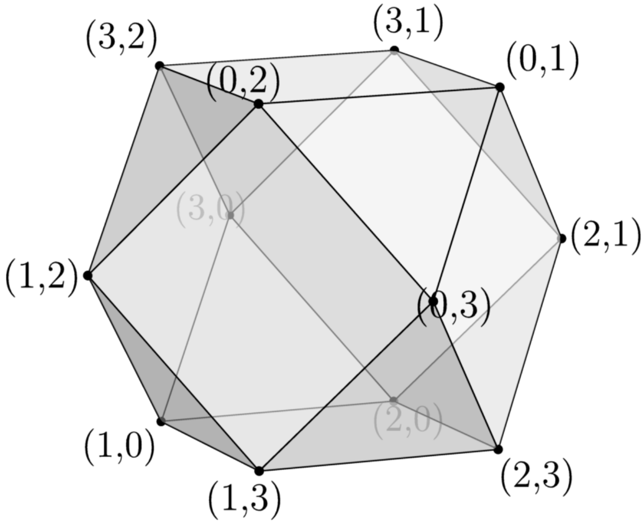
For a finite set V, Δ^V is the standard simplex with V vertices, i.e.,

$$\Delta^V = \{\lambda \in [0,1]^V : \sum_{v \in V} \lambda_v = 1\}.$$

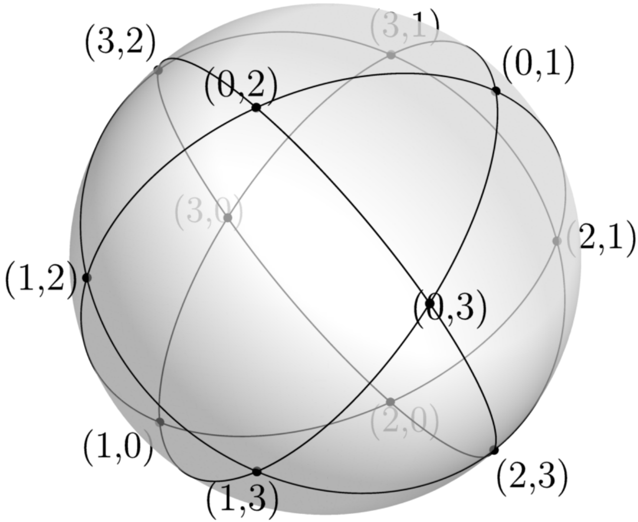
Let G be a graph, we construct a topological space $\mathrm{Bx}(G)$ as the subspace of $\Delta^{V_G} \times \Delta^{V_G}$ consisting of points (λ, ρ) such that

$$\{u : \lambda_u > 0\} \times \{v : \rho_v > 0\} \subseteq E_G.$$

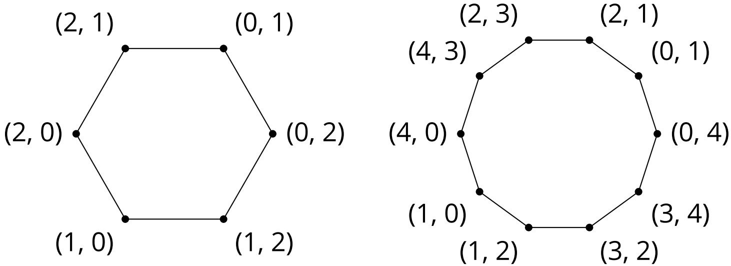
$\mathrm{Bx}(K_4)$



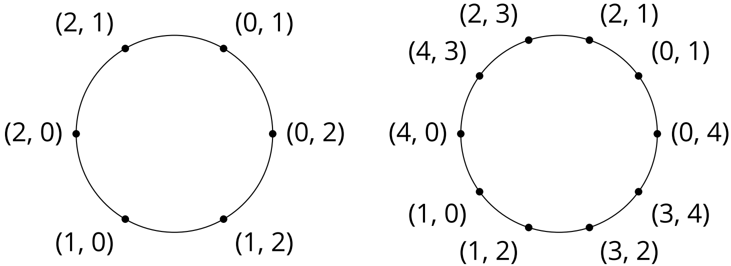
Bx(K_4)



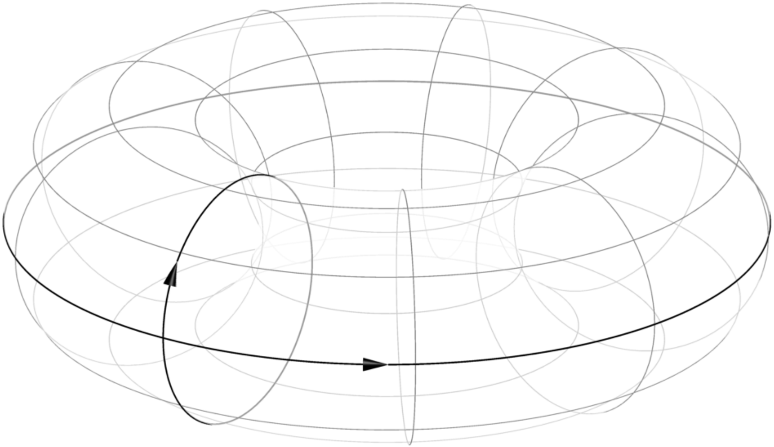
$\mathrm{Bx}(K_3)$, $\mathrm{Bx}(C_5)$, ...



$\mathrm{Bx}(K_3), \mathrm{Bx}(C_5), \ldots$



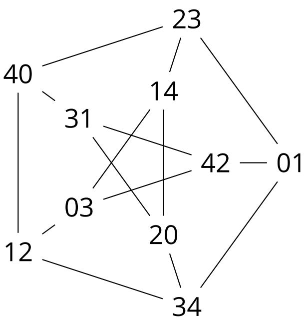
$\mathsf{Bx}(C_{2k+1}{}^{2})$*



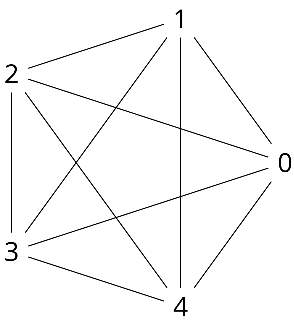
*up to homotopy equivalence

Kneser's conjecture and Lovász's proof

Kneser graph $K(k, n)$ (where $2n < k$) is the graph whose vertices are n-element subsets of $[k]$, and edges are disjoint sets.



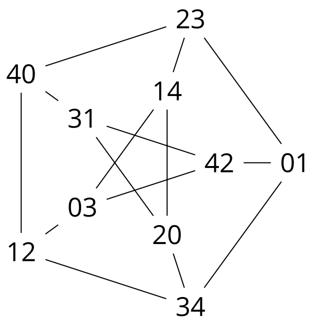
$K(5, 2)$



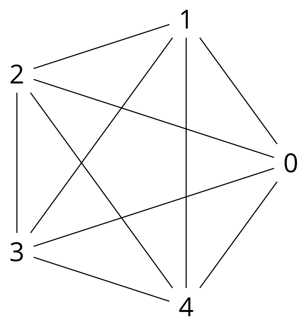
$K(5, 1)$

Kneser's conjecture and Lovász's proof

Kneser graph $K(k, n)$ (where $2n < k$) is the graph whose vertices are n-element subsets of $[k]$, and edges are disjoint sets.



$K(5, 2)$

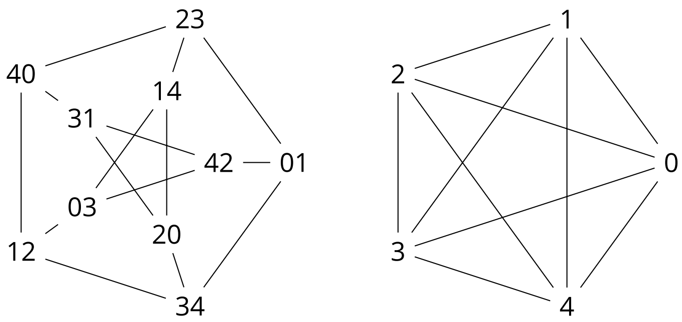


$K(5, 1)$

Kneser's conjecture. The chromatic number of $K(2n + k - 2, n)$ is k.

Kneser's conjecture and Lovász's proof

Kneser graph $K(k, n)$ (where $2n < k$) is the graph whose vertices are n-element subsets of $[k]$, and edges are disjoint sets.



$K(5, 2)$ $K(5, 1)$

Kneser's conjecture. The chromatic number of $K(2n + k - 2, n)$ is k.

$$S^{k-2} \to \mathrm{Bx}(K(2n + k - 2, n)) \not\to \mathrm{Bx}(K_{k-1}) \to S^{k-3}$$

Borsuk-Ulam Theorem. There is no continuous map $f\colon S^{k+1} \to S^k$ such that $f(-x) = -f(x)$.

$S^k = \{x \in \mathbb{R}^{k+1} \mid \|x\| = 1\}$

Chapter III: A minion homomorphism

- Start with a polymorphism

$$f\colon {C_{2k+1}}^{n} \to K_3.$$

Chapter III: A minion homomorphism

- Start with a polymorphism

$$f\colon {C_{2k+1}}^n \to K_3.$$

- Each such polymorphism induces a continuous map

$$f'\colon \mathrm{Bx}({C_{2k+1}}^n) \to \mathrm{Bx}(K_3),$$

Chapter III: A minion homomorphism

- Start with a polymorphism

$$f\colon {C_{2k+1}}^{n} \to K_3.$$

- Each such polymorphism induces a continuous map

$$f'\colon T^n \to S^1,$$

s.t., $f(-x_1, \ldots, -x_n) = -f(x_1, \ldots, x_n)$.

$T^n = (S^1)^n = S^1 \times \cdots \times S^1$
$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

Chapter III: A minion homomorphism

- Start with a polymorphism

$$f\colon {C_{2k+1}}^n \to K_3.$$

- Each such polymorphism induces a continuous map

$$f'\colon T^n \to S^1,$$

s.t., $f(-x_1, \dots, -x_n) = -f(x_1, \dots, x_n)$.

- Homotopy classes of continuous maps $T^n \to S^1$ are in 1-to-1 correspondence with linear maps

$$f_*\colon \mathbb{Z}^n \to \mathbb{Z},$$

s.t., $f(1, \dots, 1)$ is odd (Borsuk-Ulam Theorem).

$$T^n = (S^1)^n = S^1 \times \cdots \times S^1$$
$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

Chapter III: A minion homomorphism

- Start with a polymorphism

$$f\colon {C_{2k+1}}^n \to K_3.$$

- Each such polymorphism induces a continuous map

$$f'\colon T^n \to S^1,$$

s.t., $f(-x_1, \dots, -x_n) = -f(x_1, \dots, x_n)$.

- Homotopy classes of continuous maps $T^n \to S^1$ are in 1-to-1 correspondence with linear maps

$$f_*\colon \mathbb{Z}^n \to \mathbb{Z},$$

s.t., $f(1, \dots, 1)$ is odd (Borsuk-Ulam Theorem).

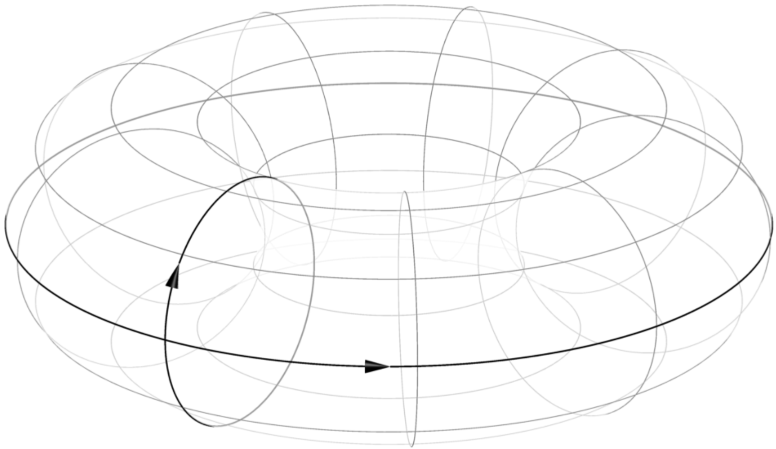
Altogether, we get a minion homomorphism:

$$\xi\colon \text{pol}(C_{2k+1}, K_3) \to \text{pol}(\mathbb{Z}).$$

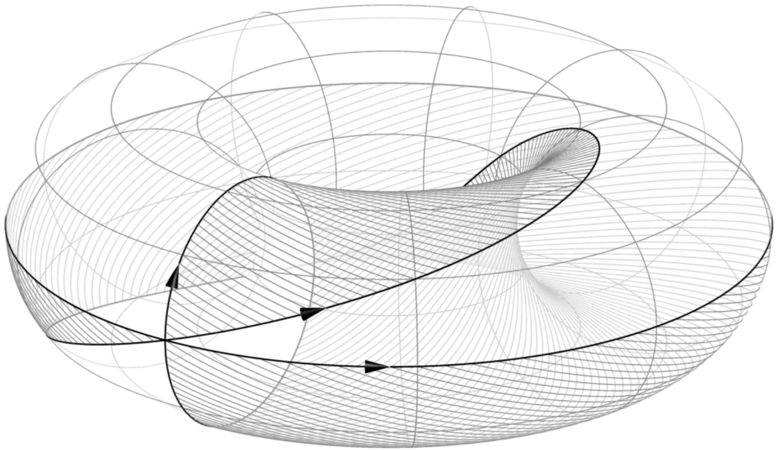
defined by $\xi(f) = f_*$.

$$T^n = (S^1)^n = S^1 \times \cdots \times S^1$$
$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

A minion homomorphism



A minion homomorphism



Finale

Corollary [Barto, Bulín, Krokhin, O., '21].

Let Γ *be a finite template promise CSP. If there is a minion homomorphism* $\xi\colon \text{pol}(\Gamma) \to \text{pol}(\mathbb{Z})$ *such that* $\xi(f) \neq 0$ *for all* $f \in \text{pol}(\Gamma)$*, then* Γ *is NP-complete.*

The minion homomorphism

$$\xi\colon \text{pol}(C_{2k+1}, K_3) \to \text{pol}(\mathbb{Z}).$$

satisfies the above.

Theorem [Krokhin, O., '19].

Colouring graphs that are promised to map homomorphically to $C_{(2k+1)}$ *with* 3 *colours is NP-hard.*

Epilogue

Theorem [Krokhin, O., '19].
Colouring graphs that are promised to map homomorphically to $C_{(2k+1)}$ *with* 3 *colours is NP-hard.*

*the proof was brought to you by [Wrochna, Živný, '20]

Krokhin, O., Wrochna, & Živný. (2023). Topology and adjunction in promise constraint satisfaction. *SIAM Journal on Computing*, 52(1), 38–79. arXiv:2003.11351, doi:10.1137/20M1378223

Epilogue

Theorem [Krokhin, O., ‘19].
Colouring graphs that are promised to map homomorphically to $C_{(2k+1)}$ *with 3 colours is NP-hard.*

*the proof was brought to you by [Wrochna, Živný, ‘20]

Krokhin, O., Wrochna, & Živný. (2023). Topology and adjunction in promise constraint satisfaction. *SIAM Journal on Computing*, 52(1), 38–79. arXiv:2003.11351, doi:10.1137/20M1378223

Theorem [Filakovský, Nakajima, O., Tasinato, Wagner, STACS‘24].
Linearly ordered 4-colouring of 3-colourable 3-uniform hypergraphs is NP-hard.

Filakovský, Nakajima, O., Tasinato, & Wagner. (2024). Hardness of linearly ordered 4-colouring of 3-colourable 3-uniform hypergraphs. *STACS 2024*.

Epilogue

Theorem [Krokhin, O., '19].
Colouring graphs that are promised to map homomorphically to $C_{(2k+1)}$ with 3 colours is NP-hard.

*the proof was brought to you by [Wrochna, Živný, '20]

Krokhin, O., Wrochna, & Živný. (2023). Topology and adjunction in promise constraint satisfaction. *SIAM Journal on Computing*, 52(1), 38–79. arXiv:2003.11351, doi:10.1137/20M1378223

Theorem [Filakovský, Nakajima, O., Tasinato, Wagner, STACS'24].
Linearly ordered 4-colouring of 3-colourable 3-uniform hypergraphs is NP-hard.

Filakovský, Nakajima, O., Tasinato, & Wagner. (2024). Hardness of linearly ordered 4-colouring of 3-colourable 3-uniform hypergraphs. *STACS 2024*.

■