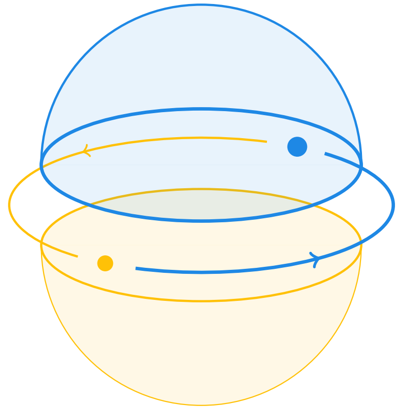
Homotopy and the complexity of homomorphism problems

Jakub Opršal et al.



joint work with Sebastian Meyer (TU Dresden)

UNIVERSITY OF BIRMINGHAM

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Bulatov–Zhuk Theorem **[Bulatov, 2017; Zhuk, 2017]**
For every finite structure A, CSP(A) is NP-complete or in P.

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Fact. There is no **FO**-sentence ϕ in the language of graphs such that

$$\text{a graph } G \text{ is } 2\text{-colourable} \Leftrightarrow G \models \phi$$

(Proof via Ehrenfeucht-Fraïssé games.)

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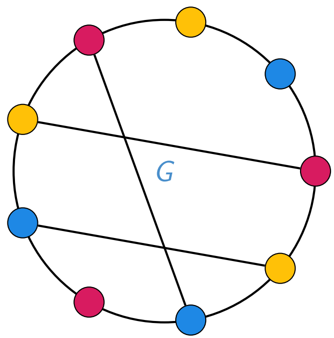
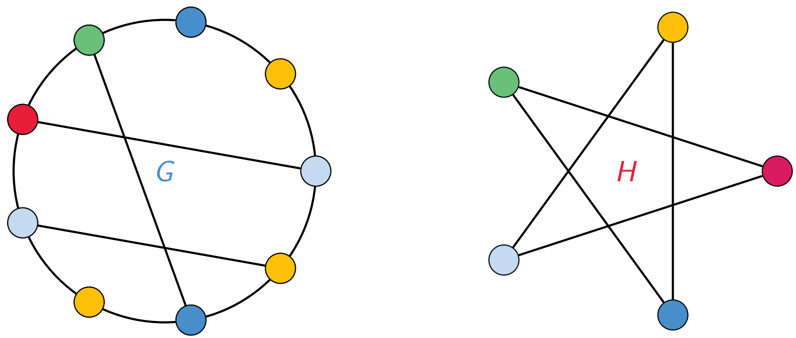
Why is homotopy theory so effective in computational complexity of CSPs?

Part I. What problems am I talking about?

Graph **colouring**



Graph colouring



Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, a graph homomorphism $G \to H$ is a mapping $h\colon V_G \to V_H$ that preserves edges,

$$uv \in E_G \Rightarrow h(u)h(v) \in E_H.$$

Example. A colouring of a graph G with k colours is just a homomorphism $c\colon G \to K_k$.

The H-colouring problem

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H-colouring

Fix a graph H (called *template*). Given a graph G, decide whether there is a homomorphism $G \to H$.

- K_2-colouring is easy (it is solvable in logspace [Reingold, 2005]);
- K_k-colouring is NP-complete for all $k > 2$.
- What about other graphs H?

Theorem [Hell & Nešetřil, 1990].
Unless P = NP, the only graph H-colouring problem that is solvable in polynomial time is 2-colouring.

Outline of a new proof

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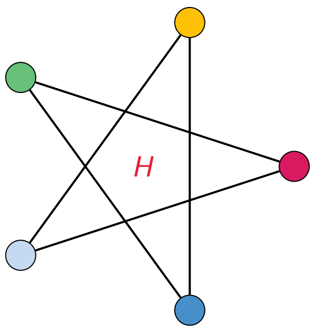

1. Identify which problems are NP-hard using the *algebraic approach to the constraint satisfaction problem*.
2. If H-colouring is not NP-hard, show that its *solution spaces* are component-wise contractible.
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Part II. What the . . . is the solution space of H-colouring?

Solution posets: Multihomomorphisms

A multihomomorphism is a function $f: V(G) \to 2^{V(H)} \setminus \{\emptyset\}$ such that, for all edges $uv \in E(G)$, we have that

$$f(u) \times f(v) \subseteq E(H).$$

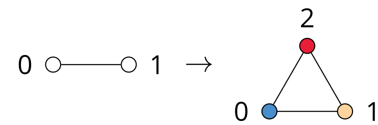
- Multihomomorphisms are naturally ordered

$$f \leq g \Leftrightarrow f(u) \subseteq g(u) \text{ for all } u$$

- $\text{mhom}(G, H)$ is the poset of multihomomorphisms.

Solution posets: Multihomomorphisms

$K_2 \to K_3$



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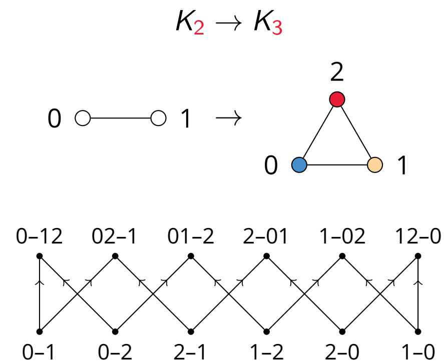

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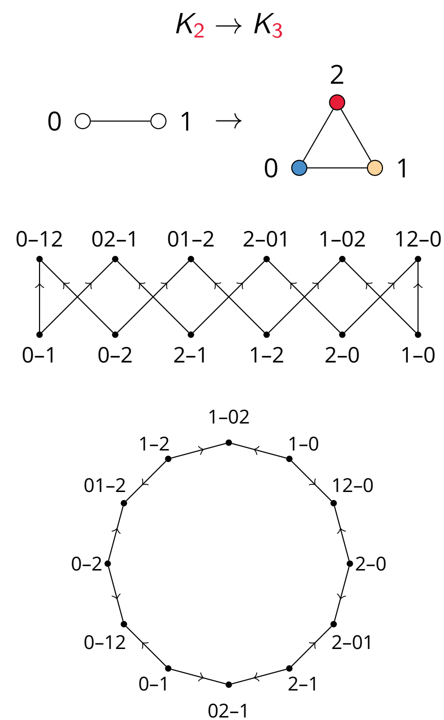


Solution spaces

Given graphs G and H, we define the space

$$\mathrm{Hom}(G, H) = |N \,\mathrm{mhom}(G, H)|$$

- The vertices are multihomomorphisms,
- f and g are connected by an arc if $f \leq g$,
- $\{f, g, h\}$ form a triangle if $f \leq g \leq h$,
- etc.

We view this as the solution space of instance G of H-colouring.

Example. $\mathrm{Hom}(K_2, K_3) \simeq S^1$.

Example. In $\mathrm{mhom}(K_2, K_4)$ we have:

$$0\text{–}1 \leq 02\text{–}1 \leq 02\text{–}13$$

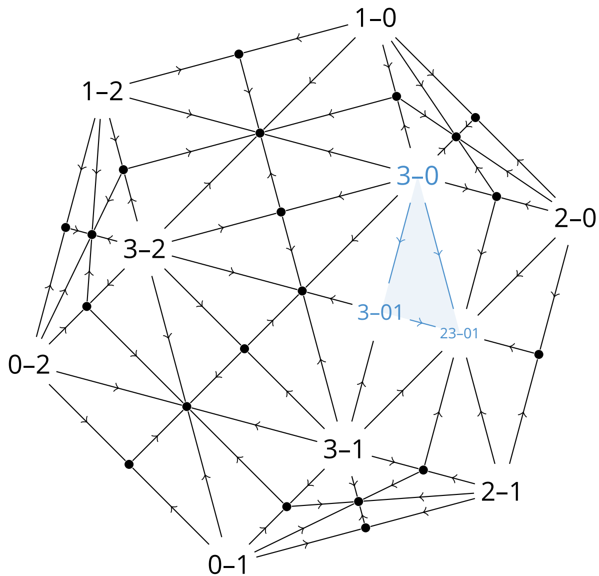
and

$$0\text{–}1 \leq 0\text{–}12 \leq 0\text{–}123$$

which creates 2-dimensional faces.

Two colourings f and g are connected if g can be obtained from f by **changing one value at a time** while remaining a valid colouring.

4-colourings of K_2



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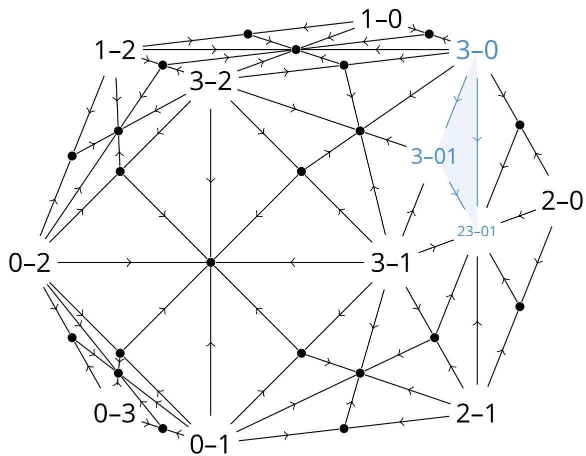

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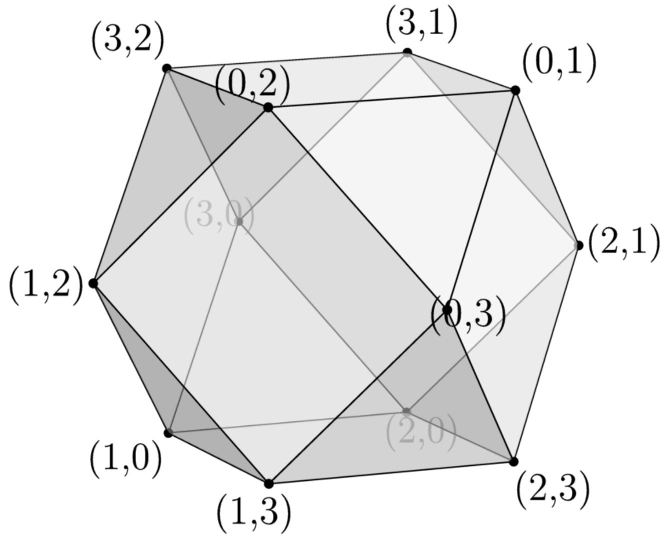

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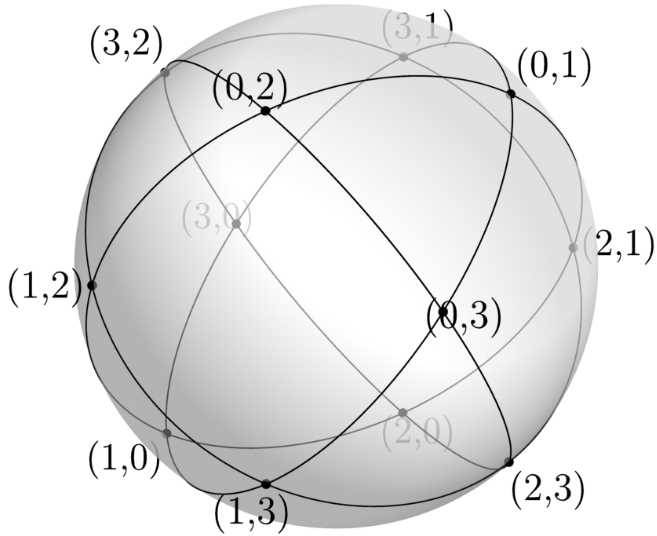

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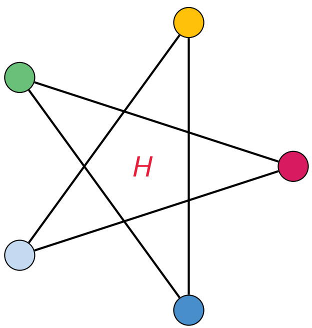
Part III. A proof

Outline of the proof

Theorem [Hell, Nešetřil, 1990].
Unless P = NP, the only graph H-colouring problem that is solvable in polynomial time is 2-colouring.



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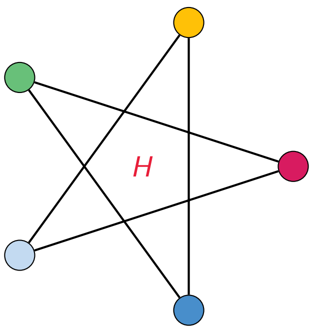

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Algebraic approach to the constraint satisfaction problem

An operation $t\colon A^n \to A$ is Taylor

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Theorem (CSP Dichotomy).

A CSP with a finite template A either

1. *admits a Taylor homomorphism $t\colon A^n \to A$, and is in P [Bulatov, 2017; Zhuk, 2017]; or*
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Lemma [Taylor, 1977].
If a topological space X admits a continuous idempotent Taylor operation t, then $\pi_1(X)$ is Abelian.

$t\colon A^n \to A$ is *idempotent* if $t(x, \dots, x) \approx x$.

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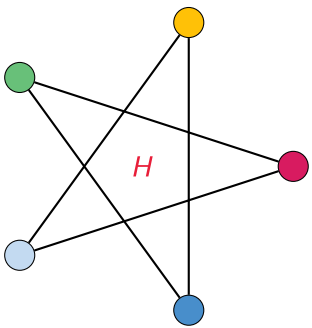

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Taylor $\to$ contractibility

A topological space X is called contractible if it is homotopy equivalent to a point $\{*\}$. For us, this is equivalent to $\pi_n(X) = 0$ for all $n \geq 0$.

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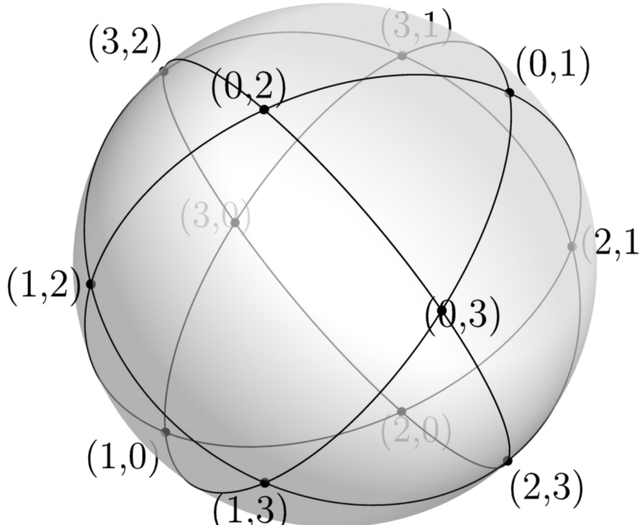
Theorem [Meyer, **O**, 2025].
Every connected finite poset that admits a monotone lax-Taylor operation is contractible, and therefore $\mathrm{Hom}(G, H)$ *is component-wise contractible for all G if H has a Taylor polymorphism.*

4-colourings of K_2



Hence, 4-colouring is NP-hard!

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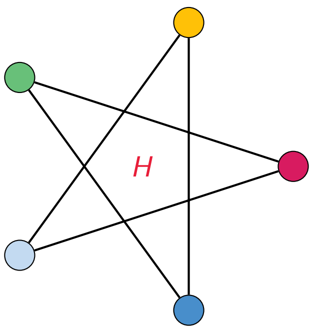

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More generally: If X is a contractible compact CW-complex, then every function $f : X \to X$ has a fixed point.

A $\mathbb{Z}_2$ action on $\mathrm{Hom}(K_2, H)$

The space $\mathrm{Hom}(K_2, H)$ admits an action of the group $\mathbb{Z}_2$, i.e., there is a homeomorphism

$$\phi\colon\ \mathrm{Hom}(K_2, H) \to \mathrm{Hom}(K_2, H)$$

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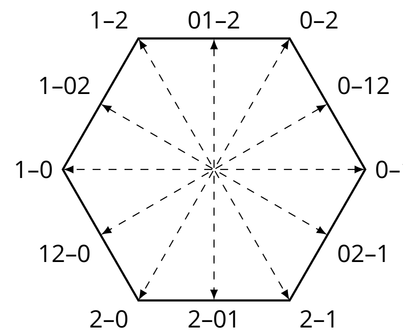

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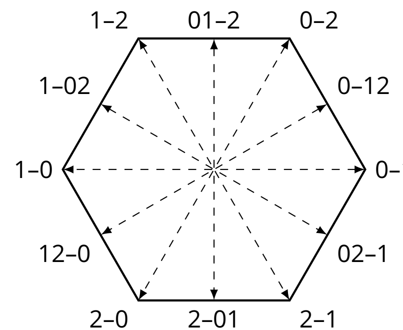

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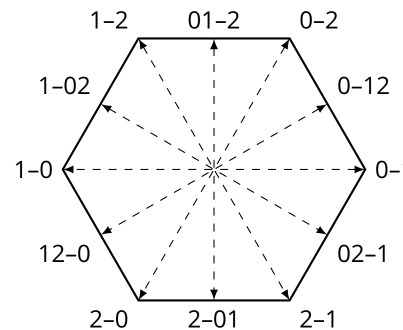
- flipping the two values induces a monotone involution on the poset, and hence a continuous involution on the space.
- ϕ does not have a fixed point, otherwise H would contain an edge uu.



A $\mathbb{Z}_2$ action on $\mathrm{Hom}(K_2, H)$

The space $\mathrm{Hom}(K_2, H)$ admits an action of the group $\mathbb{Z}_2$, i.e., there is a homeomorphism

$$\phi\colon\ \mathrm{Hom}(K_2, H) \to \mathrm{Hom}(K_2, H)$$

such that $\phi^2(x) = x$.

- ϕ is defined by flipping the two values of each multihomomorphism.
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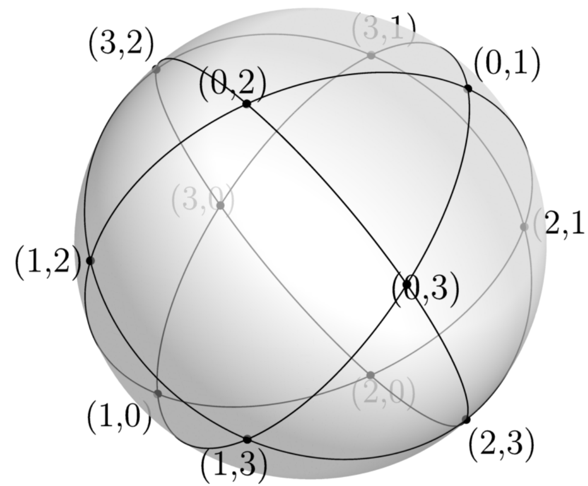

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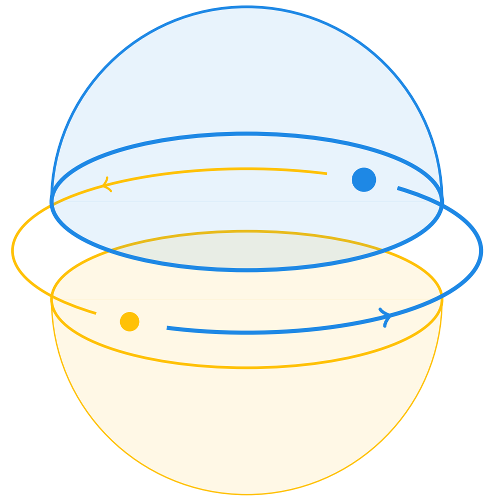
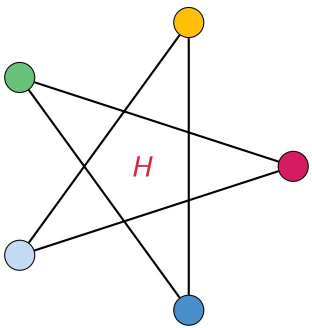


The proof

Theorem [Hell, Nešetřil, 1990].
Unless P = NP, the only graph H-colouring problem that is solvable in polynomial time is 2-colouring.



Proof. Assume that H is not-bipartite, and consider the space $X = \mathrm{Hom}(K_2, H)$.

Observe that the space admits a fixed-point free $\mathbb{Z}_2$-action $\phi\colon X \to X$ that for each multihomomorphism m flips the values of $m(0)$ and $m(1)$.

If H is not-bipartite then ϕ fixes a connected component of X. Indeed, if uv is an edge of an odd cycle of H then uv is connected to $vu = \phi(uv)$.

If H admitted a Taylor homomorphism, $\mathrm{mhom}(K_2, H)$ would admit a lax-Taylor operation, and all its connected component would be contractible.

Hence, ϕ which acts on the component of uv has a fixed point, the contradiction. ∎

How it started...

Conjecture [Brakensiek, Guruswami, 2018].

Colouring graphs that are promised to map homomorphically to $C_{(2k+1)}$ *with* c *colours is NP-complete for all* $c > 2$ *and* $k > 0$*.*

[1] Krokhin, **O** (2019). The complexity of 3-colouring H-colourable graphs. *FOCS 2019*.

[2] Wrochna, Živný (2020). Improved hardness for H-colourings of G-colourable graphs. *SODA 2020*.

[3] Avvakumov, Filakovský, **O**, Tasinato, & Wagner (2025). Hardness of 4-colouring G-colourable graphs. *STOC 2025*.

Theorem [Avvakumov et al., 2025].

Colouring graphs that are promised to map homomorphically to $C_{(2k+1)}$ *with* 4 *colours is NP-complete.*

[4] Filakovský, Nakajima, **O**, Tasinato, & Wagner (2024). Hardness of Linearly Ordered 4-Colouring of 3-Colourable 3-Uniform Hypergraphs *STACS 2024*.

How does the topology of the solution space influence the complexity of a computational problem?

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[1] Schnider, Weber (2024). A topological version of Schaefer's dichotomy theorem. *SoCG 2024*.

[2] Meyer (2024). A dichotomy for finite abstract simplicial complexes. *arXiv:2408.08199*.

[3] Meyer, **O** (2025). A topological proof of the Hell–Nešetřil dichotomy. *SODA 2025*.

Theorem [Meyer, 2024; Meyer, **O**, 2025].

A constraint satisfaction problem is NP-complete, unless each connected component of the solution space is contractible (i.e., topologically trivial).

Corollary [Hell & Nešetřil, 1990].

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Theorem [Briceño et al., 2017].

A constraint satisfaction problem is expressible in ***FO*** *if and only if the solutions spaces are either contractible or empty.*

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