Promises, constraint satisfaction, and problems

Beyond universal algebra (part I)

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overview

Part I (today)

- algebraic approach to (promise) constraint satisfaction.

Part II (tomorrow)

- beyond algebraic approach
- open problems

warning!

Everything is finite! (Well, almost.)

warning!

I will not talk about Galois connections. Sorry, Reinhard.

warning!

There are no algebras in this talk!

warning!

Definitions ahead.

an old story

- dichotomy of Boolean CSPs [Scheafer, “78]
- dichotomy of (undirected) graph CSPs [Hell, Nešetřil, “90]
- the dichotomy conjecture [Feder, Vardi, “98]
- pol-inv Galois correspondence [Cohen, Gyssens, Jeavons, “97]
- HSP closure [Bulatov, Jeavons, Krokhin, ‘05]
- Taylor implies WNU [Maróti, McKenzie, ‘08]
- algorithms given WNU polymorphisms [Bulatov, ‘17; Zhuk, ‘17]

a new story

reductions

Assume that **A** and **B** are two (finite) relational structures.

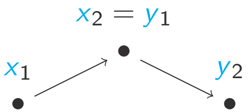
A reduction from CSP($\rho\mathbf{A}$) to CSP($\mathbf{A}$) is a mapping

$$\lambda\colon \text{structures similar to } \rho\mathbf{A} \to \text{structures similar to } \mathbf{A}$$

such that

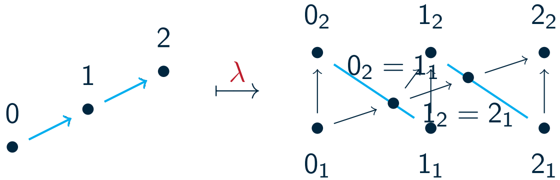
$$\mathbf{I} \to \rho\mathbf{A} \quad \text{iff} \quad \lambda\mathbf{I} \to \mathbf{A}.$$

a gadget reduction λ



$$\phi(x_1, x_2, y_1, y_2) = (x_1, x_2) \in E \wedge (y_1, y_2) \in E \wedge x_2 = y_1.$$

Example



a pp-power ρ

$\rho\mathbf{A}$ is a pp-power of $\mathbf{A}$.

Concretely, $\rho\mathbf{A} = (A^2; E^{\rho\mathbf{A}})$ where

$$((a_1, a_2), (b_1, b_2)) \in E^{\rho\mathbf{A}}$$
$$\text{iff} \quad \mathbf{A} \models \phi(a_1, a_2, b_1, b_2)$$
$$\text{iff} \quad (a_1, a_2) \in E^{\mathbf{A}} \wedge (b_1, b_2) \in E^{\mathbf{A}} \wedge a_2 = b_1 .$$

Observation

$$\mathbf{I} \to \rho\mathbf{A} \quad \text{iff} \quad \lambda\mathbf{I} \to \mathbf{A}$$

■

algebraic approach in a nutshell

Theorem [Bulatov, Jeavons, Krokhin, '05; Barto, **O**, Pinsker, '17]
The following are equivalent for any finite relational structures $\mathbf{A}$, $\mathbf{B}$:

1. there is a gadget reduction from $\mathrm{CSP}(\mathbf{B})$ to $\mathrm{CSP}(\mathbf{A})$;
2. $\mathbf{B}$ is homomorphically equivalent to a pp-power of $\mathbf{A}$;
3. there is a minion (h1 clone) homomorphism from $\mathrm{pol}(\mathbf{A})$ to $\mathrm{pol}(\mathbf{B})$.

promises

definition of promise contraint satisfaction

Fix two finite relational structures $\mathbf{A}, \mathbf{B}$ in the same finite language with a homomorphism $\mathbf{A} \to \mathbf{B}$.

PCSP($\mathbf{A}, \mathbf{B}$) is the following problem:

Search

Given a finite structure $\mathbf{I}$ that maps homomorphically to $\mathbf{A}$,
find a homomorphism $h\colon \mathbf{I} \to \mathbf{B}$.

Decide

Given $\mathbf{I}$ arbitrary structure with the same language,

- accept if $\mathbf{I} \to \mathbf{A}$,
- reject if $\mathbf{I} \not\to \mathbf{B}$.

example: 1in3- vs. NAE-Sat

- 1in3-Sat is a CSP with the template $\mathbf{T}_2 = (\{0,1\}; \mathbf{1}\text{-in-}\mathbf{3})$ where $\mathbf{1}\text{-in-}\mathbf{3} = \{(0,0,1),(0,1,0),(1,0,0)\}$.
- NAE-Sat is a CSP with the template $\mathbf{H}_2 = (\{0,1\}; \mathrm{nae}_2)$ where $\mathrm{nae}_2 = \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$.

Clearly, $\mathbf{1}\text{-in-}\mathbf{3} \subseteq \mathrm{nae}_\mathbf{2}$, and therefore $\mathbf{T}_2 \to \mathbf{H}_2$.

The goal here is, given a solvable instance $\mathbf{I}$ of 1in3-Sat, find a solution to $\mathbf{I}$ as a NAE-Sat instance.

Both 1in3-Sat and NAE-Sat are NP-complete, but $\mathrm{PCSP}(\mathbf{T}_2, \mathbf{H}_2)$ is in P [Brakensiek, Guruswami, ‘16].

reductions of promise problems

A reduction from $\mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2)$ to $\mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$ is a mapping λ: such that

$$\mathbf{I} \to \mathbf{B}_1 \Rightarrow \lambda\mathbf{I} \to \mathbf{A}_1$$
$$\mathbf{I} \to \mathbf{B}_2 \Leftarrow \lambda\mathbf{I} \to \mathbf{A}_2.$$

Example

Assuming λ is the identity (do nothing):

$$\mathbf{I} \to \mathbf{B}_1 \Rightarrow \mathbf{I} \to \mathbf{A}_1 \quad \text{iff} \quad \mathbf{B}_1 \to \mathbf{A}_1$$
$$\mathbf{I} \to \mathbf{B}_2 \Leftarrow \mathbf{I} \to \mathbf{A}_2 \quad \text{iff} \quad \mathbf{B}_2 \leftarrow \mathbf{A}_2.$$

Definition. We say that $(\mathbf{B}_1, \mathbf{B}_2)$ is a homomorphic relaxation of $(\mathbf{A}_1, \mathbf{A}_2)$ if $\mathbf{B}_1 \to \mathbf{A}_1$ and $\mathbf{A}_2 \to \mathbf{B}_2$.

reductions of promise problems

A reduction from $\operatorname{PCSP}(\mathbf{B}_1, \mathbf{B}_2)$ to $\operatorname{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$ is a mapping λ: such that

$$\mathbf{I} \to \mathbf{B}_1 \Rightarrow \lambda\mathbf{I} \to \mathbf{A}_1$$
$$\mathbf{I} \to \mathbf{B}_2 \Leftarrow \lambda\mathbf{I} \to \mathbf{A}_2.$$

Example

Assuming λ is a gadget replacement, we have (for $i = 1, 2$)

$$\mathbf{I} \to \rho\mathbf{A}_i \Leftrightarrow \lambda\mathbf{I} \to \mathbf{A}_i$$

Therefore λ is a reduction from $\operatorname{PCSP}(\mathbf{B}_1, \mathbf{B}_2)$ to $\operatorname{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$ iff $\mathbf{B}_1 \to \rho\mathbf{A}_1$ and $\rho\mathbf{A}_2 \to \mathbf{B}_2$.

Definition. We say that $(\rho\mathbf{A}_1, \rho\mathbf{A}_2)$ is a pp-power of $(\mathbf{A}_1, \mathbf{A}_2)$.

Theorem ([Barto, Bulín, Krokhin, **O**, ’19])

The following are equivalent for finite structures $\mathbf{A}_{1,2}, \mathbf{B}_{1,2}$*:*

1. *there is a gadget reduction from* $\mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2)$ *to* $\mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$*;*
2. $(\mathbf{B}_1, \mathbf{B}_2)$ *is a homomorphic relaxation of a pp-power of* $(\mathbf{A}_1, \mathbf{A}_2)$*;*
3. *???!*

the best gadget reduction

$$\mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2) \xrightarrow{\lambda_1} \mathrm{PCSP}(\mathscr{P}, ?_{\mathbf{B}}) \xrightarrow{\mathrm{id}} \mathrm{PCSP}(\mathscr{P}, ?_{\mathbf{A}}) \xrightarrow{\lambda_2} \mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$$

Both λ_1 and λ_2 are essentially ‘gadget reductions’. I will also describe the corresponding ‘pp-powers’.

- λ_1 and ρ_1, so that

$$\mathbf{I} \to \rho_1 \mathscr{M} \iff \lambda_1 \mathbf{I} \to \mathscr{M}$$

- λ_2 and ρ_2, so that

$$\Sigma \to \rho_2 \mathbf{A} \iff \lambda_2 \Sigma \to \mathbf{A}$$

formulation of CSP($\mathscr{P}$)

Problem

Given a minor (strong Mal'cev) condition Σ, decide whether Σ is trivial, i.e., satisfied by projections on a set of size at least 2.

A minor condition is a finite set of identities of the form

$$f(x_{\pi(1)}, \dots, x_{\pi(n)}) \approx g(x_1, \dots, x_m)$$

for some $\pi \colon [n] \to [m]$. We often use a shorthand $f^\pi \approx g$ for the above.

ρ_2: polymorphisms

We say that $f\colon A_1^n \to A_2$ is a polymorphism from $\mathbf{A}_1$ to $\mathbf{A}_2$ of arity n if one of the following equivalent conditions is satisfied:

- f is a homomorphism from $\mathbf{A}_1^n$ to $\mathbf{A}_2$,
- for each relation $R^{\mathbf{A}_1}$ and all tuples $\mathbf{a}_1, \dots, \mathbf{a}_n \in R^{\mathbf{A}_1}$ we have

$$f(\mathbf{a}_1, \dots, \mathbf{a}_n) \in R^{\mathbf{A}_2}.$$

The set of all such polymorphisms of arity n is denoted by $\operatorname{pol}^{(n)}(\mathbf{A}_1, \mathbf{A}_2)$, and $\operatorname{pol}(\mathbf{A}_1, \mathbf{A}_2) = \bigcup_{n\in\mathbb{N}} \operatorname{pol}^{(n)}(\mathbf{A}_1, \mathbf{A}_2)$.

ρ_2: polymorphisms

If $f \in \text{pol}^{(n)}(\mathbf{A}_1, \mathbf{A}_2)$ and $\pi \colon [n] \to [m]$, then

$$f^\pi \colon (x_1, \dots, x_n) \mapsto f(x_{\pi(1)}, \dots, x_{\pi(n)}) \in \text{pol}^{(m)}(\mathbf{A}_1, \mathbf{A}_2).$$

The function f^π is called the minor of f defined by π.

A non-empty set of functions from a set A_1 to a set A_2 that is closed under taking minors is called a function minion.

- any (function) clone is a function minion, $\mathscr{P}$ is the projection minion.
- we say that a minor condition Σ is satisfied in $\mathscr{M}$ (and write $\Sigma \to \mathscr{M}$) if there is $\xi \colon \Sigma \to \mathscr{M}$ s.t.

$$\xi(f)^\pi = \xi(g) \text{ for each identity } f^\pi \approx g.$$

$\lambda_2\colon \mathrm{PCSP}(\mathscr{P}, \mathscr{M}) \to \mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$

Given a minor condition Σ, construct an instance $\mathbf{I}_{\mathbf{A}_1}(\Sigma)$ of PCSP($\mathbf{A}_1, \mathbf{A}_2$):

- for each symbol f of arity n in Σ, take a copy of $\mathbf{A}_1^n$ with vertices labelled by $f(a_1, \dots, a_n)$ for $a_{1,\dots,n} \in \mathbf{A}_1$;
- for each identity

$$f(x_{\pi(1)}, \dots, x_{\pi(n)}) \approx g(x_1, \dots, x_m)$$

where $\pi\colon [n] \to [m]$, and $a_{1,\dots,m} \in \mathbf{A}_1$, identify vertices labelled

$$f(a_{\pi(1)}, \dots, a_{\pi(n)}) \text{ and } g(a_1, \dots, a_m).$$

λ_2 & ρ_2: the second reduction

Observation. For all $\mathbf{C}$, we have

$$\Sigma \to \mathrm{pol}(\mathbf{A}_1, \mathbf{C}) \iff \mathbf{I}_{\mathbf{A}_1}(\Sigma) \to \mathbf{C}.$$

Theorem

The indicator structure gives a reduction:

$$\mathrm{PCSP}(\mathscr{P}, \mathrm{pol}(\mathbf{A}_1, \mathbf{A}_2)) \xrightarrow{\mathbf{I}_{\mathbf{A}_1}} \mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$$

Proof. We need to show that

1. if Σ is trivial, then $\mathbf{I}_{\mathbf{A}_1}(\Sigma) \to \mathbf{A}_1$, and
2. if $\mathbf{I}_{\mathbf{A}_1}(\Sigma) \to \mathbf{A}_2$ then $\Sigma \to \mathrm{pol}(\mathbf{A}_1, \mathbf{A}_2)$.

(2) follows directly, but also (1) follows since $\mathscr{P} \to \mathrm{pol}(\mathbf{A}_1, \mathbf{A}_1)$. ∎

$\lambda_1\colon \mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2) \to \mathrm{PCSP}(\mathscr{P}, \mathscr{B})$

Starting with $\mathbf{I}$ similar to $\mathbf{B}_1$, construct a minor condition $\Sigma(\mathbf{B}_1, \mathbf{I})$:

- for each $v \in I$, add to Σ a symbol f_v of arity B_1,
- for each $(v_1, \dots, v_k) \in R^{\mathbf{I}}$, add to Σ a symbol $g_{(v_1,\dots,v_k),R}$ of arity $R^{\mathbf{B}_1}$, and
- introduce identities

$$f_{v_1}(x_{b_1}, \dots, x_{b_n}) \approx g_{(v_1,\dots,v_k),R}(x_{r_1(\mathbf{1})}, \dots, x_{r_m(\mathbf{1})})$$

$$\vdots$$

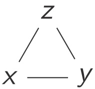
$$f_{v_\mathbf{k}}(x_{b_1}, \dots, x_{b_n}) \approx g_{(v_1,\dots,v_\mathbf{k}),R}(x_{r_1(\mathbf{k})}, \dots, x_{r_m(\mathbf{k})})$$

where $R^{\mathbf{B}_1} = \{r_i \mid i \in [m]\}$ and $B_1 = \{b_i \mid i \in [n]\}$.

examples of conditions from structures

- $\Sigma(K_3, \circlearrowleft)$ is the Siggers identity!

$$\begin{aligned} v(x, y, z) &\approx s(x, y, z, x, y, z) \\ v(x, y, z) &\approx s(y, x, x, z, z, y) \end{aligned}$$



- $\Sigma(K_3, K_3)$ is trivial!
- $\Sigma(\mathbf{T}, \circlearrowleft_3)$ is ternary weak near unanimity!
  ($\mathbf{T}$ is the template of 1in3-Sat.)
- $\Sigma(\text{1-in-}k, \text{inj}_{k,n})$,
  where $\text{inj}_{k,n} = \{(a_1, \ldots, a_k) \mid a_i \in [n], a_i \neq a_j \text{ if } i \neq j\}$,
  are (n, k) dissected weak near unanimity identities. [GJKMP'20].

ρ_1: the free structure

Given a minion $\mathscr{M}$ and a (finite) structure $\mathbf{B}_1$, we define a structure $\mathbf{F}_{\mathscr{M}}(\mathbf{B}_1)$:

- the universe are the B_1-ary functions in $\mathscr{M}$, i.e., $F_{\mathscr{M}}(\mathbf{B}_1) = \mathscr{M}^{(B_1)}$,
- the relation $R^{\mathbf{F}}$ is defined to contain all tuples $(f_1, \dots, f_k)$ such that there is $g \in \mathscr{M}^{(R^{\mathbf{B}_1})}$ and

$$f_1(x_{b_1}, \dots, x_{b_n}) \approx g(x_{r_1(1)}, \dots, x_{r_m(1)})$$
$$\vdots$$
$$f_k(x_{b_1}, \dots, x_{b_n}) \approx g(x_{r_1(k)}, \dots, x_{r_m(k)})$$

where $R^{\mathbf{B}_1} = \{r_i \mid i \in [m]\}$ and $B_1 = \{b_i \mid i \in [n]\}$.

λ_1 & ρ_1: the first reduction

Observation. for all $\mathbf{C}$, we have

$$\mathbf{C} \to \mathbf{F}_{\mathscr{M}}(\mathbf{B}_1) \iff \Sigma(\mathbf{B}_1, \mathbf{C}) \to \mathscr{M}$$

Theorem

The assignment $\mathbf{I} \mapsto \Sigma(\mathbf{B}_1, \mathbf{I})$ *gives a reduction:*

$$\mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2) \xrightarrow{\Sigma(\mathbf{B}_1, -)} \mathrm{PCSP}(\mathscr{P}, \mathrm{pol}(\mathbf{B}_1, \mathbf{B}_2))$$

back to the whole reduction

$$\mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2) \xrightarrow{\lambda_1} \mathrm{PCSP}(\mathscr{P}, \mathscr{B}) \xrightarrow{\mathrm{id}} \mathrm{PCSP}(\mathscr{P}, \mathscr{A}) \xrightarrow{\lambda_2} \mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$$

where $\mathscr{A} = \mathrm{pol}(\mathbf{A}_1, \mathbf{A}_2)$ and $\mathscr{B} = \mathrm{pol}(\mathbf{B}_1, \mathbf{B}_2)$.

To make the middle reduction work, we need

$$\mathscr{P} \to \mathscr{P} \quad \text{and} \quad \mathscr{A} \to \mathscr{B}.$$

A minion homomorphism is a mapping $\xi \colon \mathscr{M} \to \mathscr{N}$ s.t.

$$\xi(f)^\pi = \xi(f^\pi) \text{ for all } \pi \colon [n] \to [m].$$

Such homomorphisms preserve satisfaction of minor conditions.

conclusion

Theorem [Barto, Bulín, Krokhin, **O**, '19]

The following are equivalent for all pairs of similar relational structures $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{B}_1, \mathbf{B}_2$:

1. there is a gadget reduction from $\mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2)$ to $\mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$;
2. $(\mathbf{B}_1, \mathbf{B}_2)$ is a homomorphic relaxation a pp-power of $(\mathbf{A}_1, \mathbf{A}_2)$;
3. there is a minion homomorphism from $\mathrm{pol}(\mathbf{A}_1, \mathbf{A}_2)$ to $\mathrm{pol}(\mathbf{B}_1, \mathbf{B}_2)$.

conclusion

- Generalised loop conditions $\mathbf{C} \mapsto \Sigma(\mathbf{A}, \mathbf{C})$;
- Free structure $\mathscr{M} \mapsto \mathbf{F}_{\mathscr{M}}(\mathbf{A})$;
- Indicator structure $\Sigma \mapsto \mathbf{I}_{\mathbf{A}}(\Sigma)$,
- Polymorphisms $\mathbf{C} \mapsto \mathrm{pol}(\mathbf{A}, \mathbf{C})$.

Theorem [Barto, Bulín, Krokhin, **O**, '19]
For a fixed finite structure $\mathbf{A}$. The following equivalences hold for all $\mathbf{B}$ a structure, $\mathscr{M}$ a minion, and Σ minor condition.

$$\Sigma(\mathbf{A}, \mathbf{B}) \to \mathscr{M} \quad \text{iff} \quad \mathbf{B} \to \mathbf{F}_{\mathscr{M}}(\mathbf{A}) \tag{1}$$

$$\mathbf{I}_{\mathbf{A}}(\Sigma) \to \mathbf{B} \quad \text{iff} \quad \Sigma \to \mathrm{pol}(\mathbf{A}, \mathbf{B}) \tag{2}$$

credits

- pol-inv Galois correspondence [Pippenger, ‘02]
- polymorphisms in promise constraint satisfaction [Austrin, Håstad, Guruswami, ‘17]
- inclusions of function minions [Brakensiek, Guruswami, ‘18]
- h1 clone homomorphisms for CSPs [Barto, **O**, Pinsker, ‘18]
- minion homomorphisms [Barto, Bulín, Krokhin, **O**, ‘19]
- adjunctions [Wrochna, Živný, ‘20]

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