TOPOLOGICAL CHARACTERISATION OF $SD(\wedge)$ VARIETIES

Jakub Opršal

Institute of Science and Technology Austria

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BACKGROUND

TOPOLOGICAL ALGEBRAS

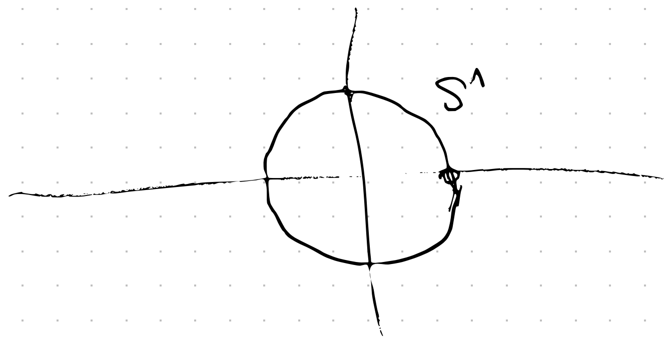
definition. A **topological algebra** $\underline{A}$ of signature σ is a topological space (A, τ) with universe A together with operations $f^{\underline{A}} : A^k \to A$ for each $f \in \sigma$ that are continuous.

examples. topological groups

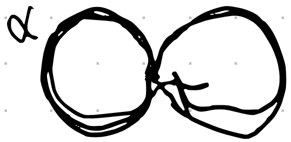
$GL(n; \mathbb{R}) \subseteq \mathbb{R}^{n\times n}$

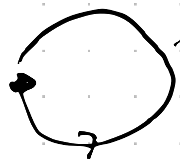
$SO(n, \mathbb{C})$, U ...

$\mathbb{C}^1$



if X topological space. $\underline{\pi_1(X)}$





$f : S^1 \longrightarrow X$ cont. up to homotopy

$g : S^1 \longrightarrow X$

theorem. For every topological group $\underline{G}$, $\pi_1(G)$ is Abelian.

theorem. (Taylor '77). Every topological Taylor algebra $\underline{A}$ has Abelian fundamental group.

Group $xy^{-1}z$ is a Taylor term.

Theorem. For every topological algebra $\underline{A}$ in an SD(1) variety,

$$\pi_n(\underline{A}) = 0$$

for all $n \geq 1$.

If X is nice : simplicial complex
& connected

~~then~~ $\pi_n(X) = 0$, $\forall n$ $\Rightarrow$ X is contractible

$X \simeq *$ up to homotopy equivalence

THE THEOREM

theorem (Taylor). Let Σ is an idempotent linear Maltsev condition, and $t \approx s$ a group law. TFAE

1/ for every ~~topo~~logical algebra $\underline{A}$ sat. Σ,
$\pi_1(\underline{A}) \models t \approx s$

2/ for every ~~topo~~logical algebra $\underline{A}$ sat. Σ,
$\pi_n(\underline{A}) \models t \approx s$ for all $n \geq 1$.

3/ every group $\mathbb{G}$ s.t. $\mathrm{Pol}(\mathbb{G}) \models \Sigma$ satisfies $t \approx s$.

$$f(x_{\sigma(1)} \ldots x_{\sigma(k)}) \approx g(x_{\rho(1)} \ldots x_{\rho(m)})$$

Polymorphisms of a group G.

$f : G^n \longrightarrow G$ homomorphism.

$(G; f \in Pol(G))$ algebra.

lemma (Taylor) If G is not abelian then $(G, Pol(G))$ does not satisfy any non-trivial idempotent Maltsev condition.

if G is abelian then $(G, Pol(G))$ has a Maltsev operation $\equiv$ ~~is~~ $\equiv$ the same of the Maltsev of G.

Pf. 3/ → 2/

> 3/ every group $\mathbb{G}$ s.t. $\mathrm{Pol}(\mathbb{G}) \models \Sigma$ satisfies $t \approx s$.
> 2/ for every topological algebra $\underline{A}$ sat. Σ, $\pi_n(\underline{A}) \models t \approx s$ for all $n \geq 1$.

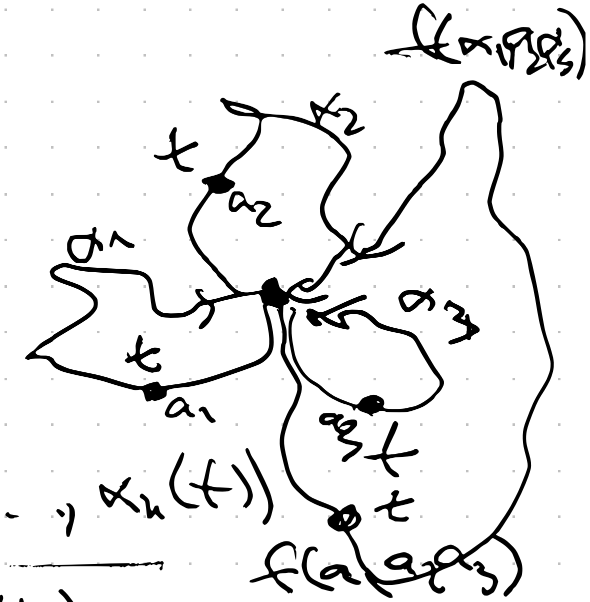
Goal. $\underline{A} \models \Sigma$ we will show that

$\mathrm{Pol}(\pi_1(A)) \models \Sigma$ ✓ $\quad m(xyx) \approx \ldots$

$f, \ldots, g$ terms of $\underline{A}$ that witness sat. Σ.

$\xi(f), \ldots, \xi(g) \in \mathrm{Pol}(\pi_1(\underline{A}))$

$\pi_1(A) := \{[\alpha] \mid \alpha : S^1 \to \underline{A}\}$



$\xi(f)(\alpha_1, \ldots, \alpha_n) : S^1 \to A$

$t \mapsto f(\alpha_1(t), \ldots, \alpha_n(t))$

$\xi(m)(\alpha\alpha\beta) : t \mapsto m(\alpha(t), \alpha(t), \beta(t)) = \alpha(t)$

Theorem (Taylor). Let Σ is an idempotent linear Maltsev condition, and $t \approx s$ a group law. TFAE

1/ for every ~~topological~~ algebra $\underline{A}$ sat. Σ,
$\pi_1(\underline{A}) \models t \approx s$

$\uparrow$ triv.

2/ for every ~~topological~~ algebra $\underline{A}$ sat. Σ,
$\pi_n(\underline{A}) \models t \approx s$ for all $n \geq 1$.

$\uparrow$ now

3/ every group $\underline{G}$ s.t. $\mathrm{Pol}(\underline{G}) \models \Sigma$ satisfies $t \approx s$.

1/ $\to$ 3/ THE CLASSIFYING SPACE

$\boxed{\begin{array}{l} \text{1/ for every topological algebra } \underline{A} \text{ sat. } \Sigma, \\ \qquad \pi_1(\underline{A}) \models t \approx s \\ \text{3/ every group } G \text{ st. } \underline{Pol(G)} \models \Sigma \\ \qquad \text{satisfies } t \approx s. \end{array}}$

Pick group G. with $Pol(G) \models \Sigma$ $f, g, \dots$

Goal. find $(\underline{A}, f, g, \dots) \models \Sigma$ topological algebra

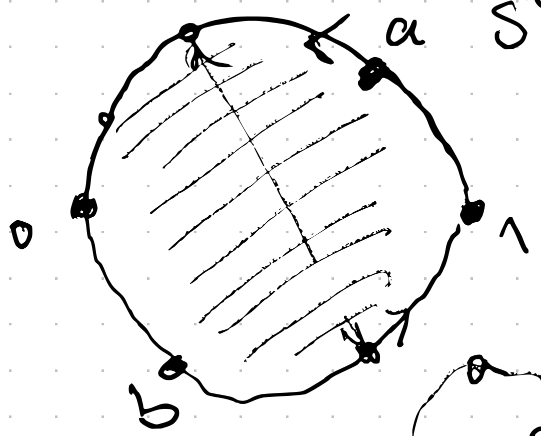
with. $\pi_1(\underline{A}) = G$.

BG is the classifying space

$\pi_n(BG) = 0$ for all $n \neq 1$.

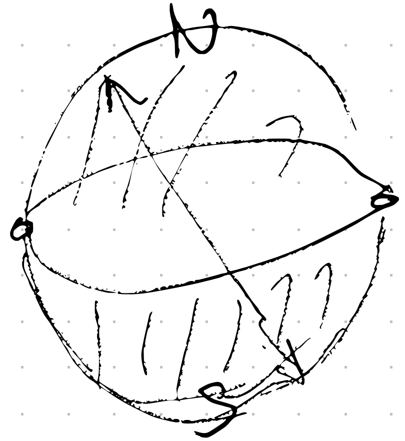
example $G = \mathbb{Z}_2$.

$B\mathbb{Z}_2 : S^\infty / \mathbb{Z}_2 \simeq \mathbb{RP}^\infty$



$$f(10) \xrightarrow{a,\ b} f(01)$$

$$S^1 \hookrightarrow S^2 \hookrightarrow S^3 \hookrightarrow \cdots \quad \underbrace{S^\infty}$$



$$\mathbb{RP}^1 \simeq S^1 \hookrightarrow \mathbb{RP}^2 \hookrightarrow$$

$\pi_1(B\mathbb{Z}_2) = \mathbb{Z}_2$

$f : \mathbb{Z}_2^\Sigma \longrightarrow \mathbb{Z}_2$

$\underline{\xi(f)} : (B\mathbb{Z}_2)^\Sigma \longrightarrow B\mathbb{Z}_2$

theorem. the following are equivalent for an idempotent (linear) variety $\mathcal{V}$:

$\pi_1(X) = 0$

$\pi_1(X) \models x = 0$

- every topological algebra in $\mathcal{V}$ has trivial homotopy,
- $\mathcal{V}$ omits types 1 & 2,
- $\mathcal{V}$ has $\wedge$-semidistributive congruence lattices.
- $\mathcal{V}$ satisfies the commutator identity $[\alpha, \beta] = \alpha \wedge \beta$.
- if $\underline{\mathrm{Pol}(G)} \in \mathcal{V}$ then G is trivial

APPLICATIONS

theorem. (Taylor). the following are equivalent for a ~~topological~~ logical algebra $\underline{A}$.

1/ $\underline{A}$ generates SD(1) variety

2/ there is a continuous majority operation on $\underline{A}$.

3/ there is a ~~totally~~ symmetric cont. operation on $\underline{A}$.

4/ ...

CSP AND BOUNDED WIDTH

theorem. Let $\mathbb{A}$ be a finite core, and $\hat{\mathcal{A}} = \mathrm{pol}(\mathbb{A})$. TFAE

1) $\mathbb{A}$ has bounded width

hard [BK14] $\uparrow\downarrow$ easy [L709]

2) $\mathbb{A}$ does not pp-construct a group.

today $\uparrow\downarrow$

3) Every* connected space X s.t. $\mathcal{A} \to \mathrm{Pol}(X)$ (minion) is contractible.

?

$PCSP(C_{odd}, K_3)$

Thank you!

Krokhin, —, Wrochna, Živný. Topology and adjunction in promise constraint satisfaction. to appear at SICOMP (2022)
arXiv: 2003.11351