Algebraic topology and constraint satisfaction

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VARIETIES OBEYING HOMOTOPY LAWS

WALTER TAYLOR

The algebraic structure of a topological algebra $\mathscr{A}$ influences its topological structure in a way which is profound but not well understood. (See § 7 below for various examples.) Here we examine this influence rather generally, and give a fairly complete analysis of one of the many forms it can take, namely, the influence of the identities of $\mathscr{A}$ on the group identities obeyed by the homotopy group (or groups of the components) of $\mathscr{A}$. For $\mathscr{V}$ a *variety* (i.e. class of algebras defined by identities), and λ a *group law*, we say that $\mathscr{V}$ *obeys* λ *in homotopy* if and only if every arc-component of every topological algebra in $\mathscr{V}$ has fundamental group obeying λ. Our investigation of this relation was inspired by the much earlier results of Schreier [**44**], who proved in 1924 that topological groups have commutative homotopy (strengthened versions are due to Cartan, Pontrjagin and Hopf), and Wallace [**52**], who proved in 1953 that topological lattices are homotopically trivial (see also [**12**] and [**8**]).

Our main theorem (3.2 below) states that $\mathscr{V}$ *obeys* λ *in homotopy if and only if every group in the idempotent reduct of* $\mathscr{V}$ *obeys* λ. As a corollary, we see that for fixed λ, "$\mathscr{V}$ obeys λ in homotopy" is a Malcev-definable (see [**46**], [**40**] or [**3**]) property of $\mathscr{V}$. The hard part of the theorem is constructing a topological algebra in $\mathscr{V}$ whose fundamental group may fail to obey λ. We do this via

Theorem [Taylor, '77]

If a topological space X has a Taylor polymorphism, then $\pi_n(X)$ are Abelian for all $n > 0$.

A polymorphism of a topological space X is a continuous map $X^n \to X$, a polymorphism a group $\mathbf{G}$ is a group homomorphism $\mathbf{G}^n \to \mathbf{G}$, etc.

Theorem [Taylor, '77]

The following are equivalent for any group identity $t \approx s$ and a linear idempotent Maltsev condition Σ:

1. If pol(X) satisfies Σ then $\pi_1(X) \models t \approx s$.
2. If pol(X) satisfies Σ then $\pi_n(X) \models t \approx s$ for all $n > 0$.
3. If pol$(\mathbf{G})$ satisfies Σ then $\mathbf{G} \models t \approx s$.

Sketch of a proof

Lemma

For all topological spaces X, and all $n > 0$, there is a minion homomorphism $\mathrm{pol}^{id}(X) \to \mathrm{pol}^{id}(\pi_n(X))$.

A minion homomorphism is a mapping $\xi \colon \mathscr{M} \to \mathscr{N}$ that preserves taking minors, i.e., for all $f \in \mathscr{M}^{(n)}$ and $\pi \colon [n] \to [m]$,

$$\xi(f)(x_{\pi(1)}, \dots, x_{\pi(n)}) \approx \xi(f(x_{\pi(1)}, \dots, x_{\pi(n)})).$$

Lemma [Wrochna, Živný, '19]

If a functor Γ preserves products then there is a minion homomorphism

$$\mathrm{pol}(A, B) \to \mathrm{pol}(\Gamma A, \Gamma B)$$

for all A, B.

Sketch of a proof

Theorem

The following are equivalent for any group identity $t \approx s$ and a linear idempotent Maltsev condition Σ:

1. *If* $\mathrm{pol}(X)$ *satisfies* Σ *then* $\pi_1(X) \models t \approx s$.
2. *If* $\mathrm{pol}(X)$ *satisfies* Σ *then* $\pi_n(X) \models t \approx s$ *for all* $n > 0$.
3. *If* $\mathrm{pol}(\mathbf{G})$ *satisfies* Σ *then* $\mathbf{G} \models t \approx s$.

The previous slide shows (3→2). (2→1) is trivial.

Lemma (1→3)

There is a functor $B\colon \mathbf{Grp} \to \mathbf{Top}$ *such that* $\pi_1(B\mathbf{G}) = \mathbf{G}$, *and it preserves products!*

Promise constraint satisfaction

Fix two finite relational structures $\mathbb{A}, \mathbb{B}$ in the same finite language with a homomorphism $\mathbb{A} \to \mathbb{B}$.

PCSP($\mathbb{A}, \mathbb{B}$) (search)

Given a finite structure $\mathbb{I}$ that maps homomorphically to $\mathbb{A}$, find a homomorphism $h \colon \mathbb{I} \to \mathbb{B}$.

We will talk about PCSP(C_{2k+1}, K_3).

Conjecture [Brakensiek, Guruswami, '16]

PCSP(H, K_c) is NP-complete for any non-bipartite loopless H and any c such that H is c-colourable.

The goal

A polymorphism from $\mathbb{A}$ to $\mathbb{B}$ is a homomorphism $\mathbb{A}^n \to \mathbb{B}$. The set of all polymorphisms $\mathrm{pol}(\mathbb{A}, \mathbb{B})$ form a function minion.

Theorem [Austrin, Håstad, Guruswami, '17; Barto, Bulín, Krokhin, __, '21]
If $\mathrm{pol}(\mathbb{A}, \mathbb{B})$ allows a minion homomorphism to a minion of bounded essential arity, then $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is NP-hard.

A way

$$\textbf{Graph} \to \textbf{hTop} \to \textbf{Grp}$$

A functor **Graph** $\to$ **hTop** that preserves products* [Kozlov, '08]

For a set V, Δ^V is the standard simplex with V vertices, i.e.,

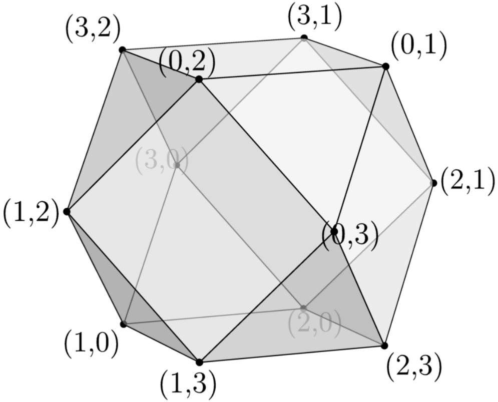
$$\Delta^V = \{\lambda \in [0,1]^V : \sum_v \lambda_v = 1\}.$$

Let G be a graph, we construct a topological space $\mathrm{Bx}(G)$ as the subspace of $\Delta^{V(G)} \times \Delta^{V(G)}$ consisting of points (λ, ρ) such that

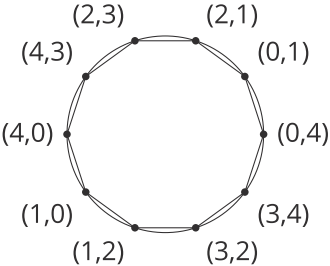
$$\{u : \lambda_u > 0\} \times \{v : \rho_v > 0\} \subseteq E(G).$$

* up to homotopy equivalence

Bx(K_4)



$\text{Bx}(C_5)$ and $\text{Bx}(K_3)$



The final piece

We compose two minion homomorphisms:

$$\mathrm{pol}(C_{2k+1}, K_3) \xrightarrow{\mathsf{Bx}} \mathrm{pol}(S^1, S^1) \xrightarrow{\pi_1} \mathrm{pol}(\mathbb{Z})$$

To get $\xi\colon \mathrm{pol}(C_{2k+1}, K_3) \to \mathrm{pol}(\mathbb{Z})$.

Lemma

If $\mathscr{M}$ is a locally finite minion† *and $\xi\colon \mathscr{M} \to \mathrm{pol}(\mathbb{Z})$ is a minion homomorphism then the image of $\mathscr{M}$ under ξ has bounded essential arity.*

† a minion $\mathscr{M}$ is locally finite if $\mathscr{M}^{(n)}$ is finite for all n.

The result

Theorem [Krokhin, __, Wrochna, Živný, '19]

For each $k > 0$, it is NP-hard to find a 3-colouring of a graph that maps to C_{2k+1}.

[1] Andrei Krokhin, __, *The complexity of 3-colouring H-colourable graphs.* FOCS, 2019.

[2] Marcin Wrochna and Stanislav Živný. *Improved hardness for H-colourings of G-colourable graphs*. SODA, 2020.

[3] Andrei Krokhin, __, Marcin Wrochna, Stanislav Živný. *Topology and adjunction in promise constraint satisfaction*. arXiv:2003.11351, 2020.

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