Datalog reductions between constraint satisfaction problems

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European Research Council
Established by the European Commission

This project has received funding under the European Union's Horizon 2020 research and innovation programme (ERC grant agreement No 714532 & MSCA grant agreement No 101034413).

Part I

Why do we care about reductions?

A reduction from a problem A to a problem B is an (efficiently computable) function ϕ that maps instances of A to instances of B and preserves the answer, i.e.,

- if $i \in A$ then $\phi(i) \in B$, and
- if $i \notin A$ then $\phi(i) \notin B$.

the class NP under P-time reductions

if P=NP

if P≠NP

the constraint satisfaction problem(s)

CSP Given a list of constraints over some domain D involving variables from V where each constraint is of the form $(v_1, \dots, v_k) \in R$ for some $R \subseteq D^k$, decide whether there is a satisfying assignment $V \to D$.

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examples

- CSP(K_3) is the 3-colouring.
- 3-SAT is expressible as CSP($\mathbf{S}_3$) for a suitable $\mathbf{S}_3$.
- Solving systems of linear equations mod p is CSP($\mathbb{Z}_p$).

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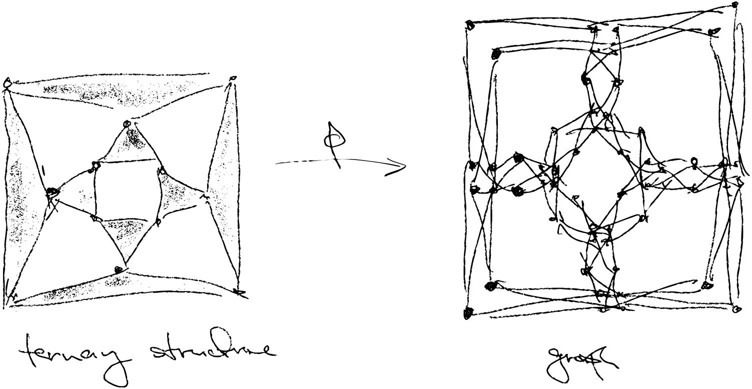
In fact, the algebraic approach is a characterisation of gadget reductions.

Theorem [Bulatov, Jeavons, & Krokhin '05 and Barto, __, & Pinsker '17].

$$\mathrm{CSP}(\mathbf{A}) \leq_{\text{gadget}} \mathrm{CSP}(\mathbf{B}) \quad \text{iff} \quad \mathrm{pol}(\mathbf{B}) \to \mathrm{pol}(\mathbf{A})$$

a gadget reduction



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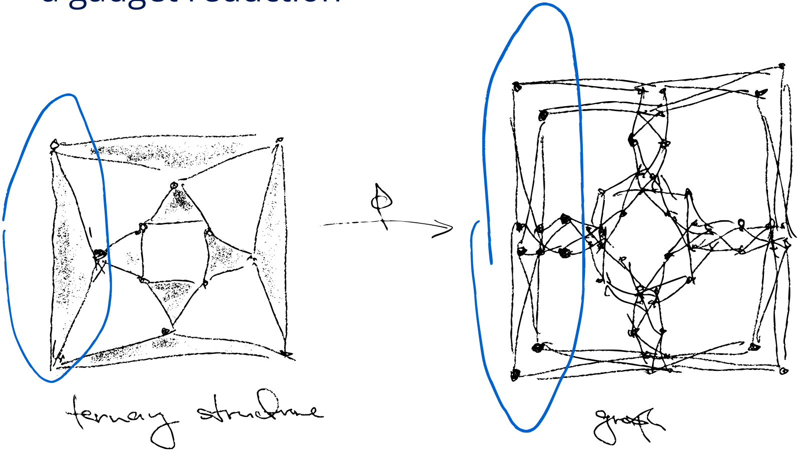
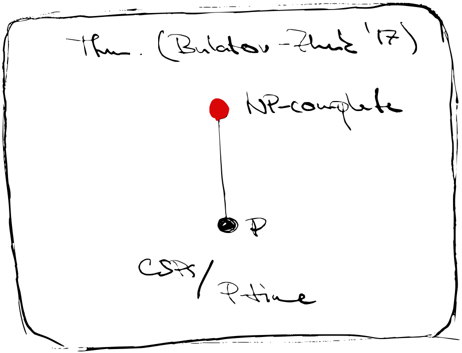
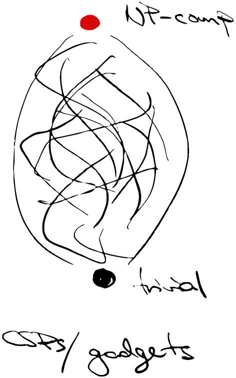


the success of algebraic approach





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PCSP($\mathbf{A}$, $\mathbf{B}$) Fix relational structures $\mathbf{A} \to \mathbf{B}$. Given a relational structure $\mathbf{Q}$ of the same type, decide between two cases:

- there is a homomorphism $h\colon \mathbf{Q} \to \mathbf{A}$, or
- there is **no** homomorphism $h'\colon \mathbf{Q} \to \mathbf{B}$.

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Not all NP-hardness of PCSPs is shown by gadget reductions, e.g., $\mathrm{PCSP}(K_3, K_5)$ is NP-hard, but

$$\text{3-SAT} \not\leq_{\text{gadget}} \mathrm{PCSP}(K_3, K_5)$$

Part II

The reduction

Datalog programs

Datalog program ϕ with input signature τ is a finite set of rules of the form

$$R(x_1, \dots, x_n) \leftarrow S_1(x_{i_1}, \dots, x_{i_{k_1}}), \dots, S_r(x_{i_{*+1}}, \dots, x_{i_{*+k_r}})$$

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For a τ-structure $\phi(\mathbf{A})$ is then computed as follows:

1. initialise: $R^\tau = R^\mathbf{A}$ if $R \in \tau$ and $R^\tau = \emptyset$ otherwise.
2. repeat until stabilises: whenever for some $x_1, \ldots, x_k \in A$ match the body, add $(x_1, \ldots, x_n)$ into R.
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Datalog can be viewed as a fragment of $\exists^+ \mathcal{L}^k_{\infty,\omega}$.

local reductions

Datalog interpretation. Fix a signature σ, a τ-structure $\mathbf{A}$, and Datalog programs ϕ and ϕ_R for $R \in \sigma$ of arities m and $m\,\mathrm{ar}(R)$ for $R \in \sigma$.

$$\mathbf{B} = (\phi(\mathbf{A}); \phi_R(\mathbf{A}), \dots)$$

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Example. $\mathrm{CSP}(\mathbf{K}_2)$ locally reduces to $\mathrm{CSP}(\mathbf{T})$ where $\mathbf{T} = (\{*\}; \bot)$ with $\bot$ being the nullary empty relation.

k-consistency as a reduction

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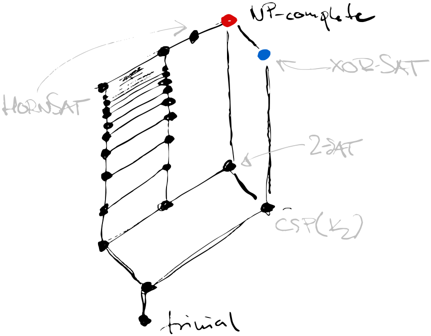
3. create the output instance $\phi(\mathbf{Q})$ of CSP($\mathbf{B}$):
   - for each K, introduce to $\phi(\mathbf{Q})$ a copy of $\mathbf{B}^{\mathcal{F}_K}$.
   - for each $L \subset K$, identify each element $b\colon \mathcal{F}_L \to B$ of $\mathbf{B}^{\mathcal{F}_K}$ with the element b' of $\mathbf{B}^{\mathcal{F}_K}$ defined as $b'(h) = b(h|_L)$.

Part III

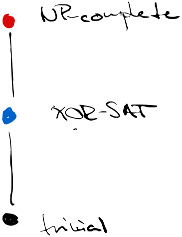
What can we prove?

Boolean CSPs (i.e., CSP($\mathbf{B}$) where the domain of $\mathbf{B}$ is $\{0, 1\}$)



[Bodirsky, Vucaj, '20]

/gadget



/gadget + Datalog

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- 3-SAT is not locally reducible to CSP($\mathbb{Z}_p$) for any prime p.

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- We have a characterisation of arc-consistency reduction by the means of certain co-monad μ acting on polymorphisms:

Theorem
$\text{CSP}(\mathbf{A})$ *reduces to* $\text{CSP}(\mathbf{B})$ *by the arc-consistency reduction iff*

$$\mu(\text{pol}(\mathbf{B})) \to \text{pol}(\mathbf{A}).$$

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Conjecture.

For all finite structures $\mathbf{A}$, if 3-SAT $\not\leq_{\text{gadget}}$ CSP($\mathbf{A}$), then

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