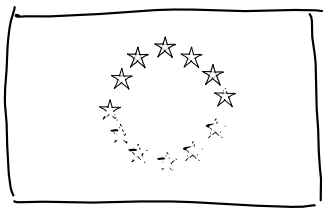
# AN INVITATION TO PROMISE CONSTRAINT SATISFACTION

Jakub Opršal

ISTA (now) → Birmingham (October)



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Resources & Co-resources

Cambridge 18/7/23

# CSP

## Constraint satisfaction problem

Can you satisfy a bunch of constraints simultaneously?

$$x + 2y \leq 2$$
$$y + z + w \leq 1$$
$$-3x - 4w \leq -3$$

## Satisfaction of primitive positive formula

Fix $\mathbb{B} = (B; R, S, \ldots)$

$\exists x_1, \ldots, x_k$ s.t.

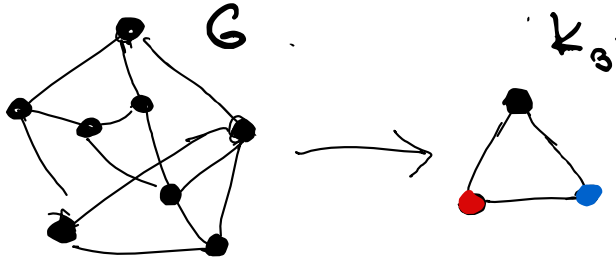
$$R(x_2, x_3) \wedge S(x_1, x_2, x_4) \wedge R(x_4, x_2) \wedge x_2 = x_3$$

## Conjunctive queries in databases

## Homomorphism problem

Fix $\mathbb{B}$, input: $\mathbb{A}$

$$\mathbb{A} \xrightarrow{?} \mathbb{B}$$




3-colouring

3SAT

lin3SAT

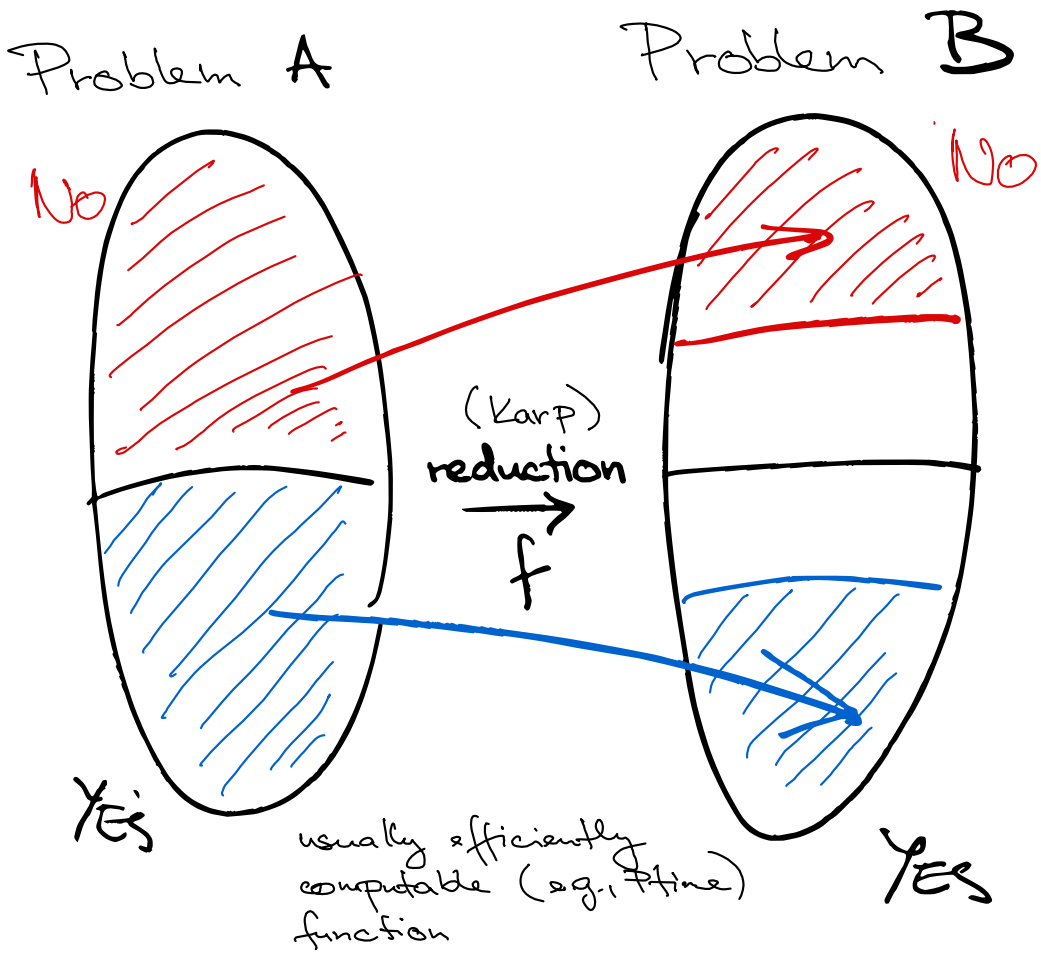
$\mathbb{Z}_2$-affine systems (XOR-SAT)

2-colouring

Linear programming

Horn-3SAT

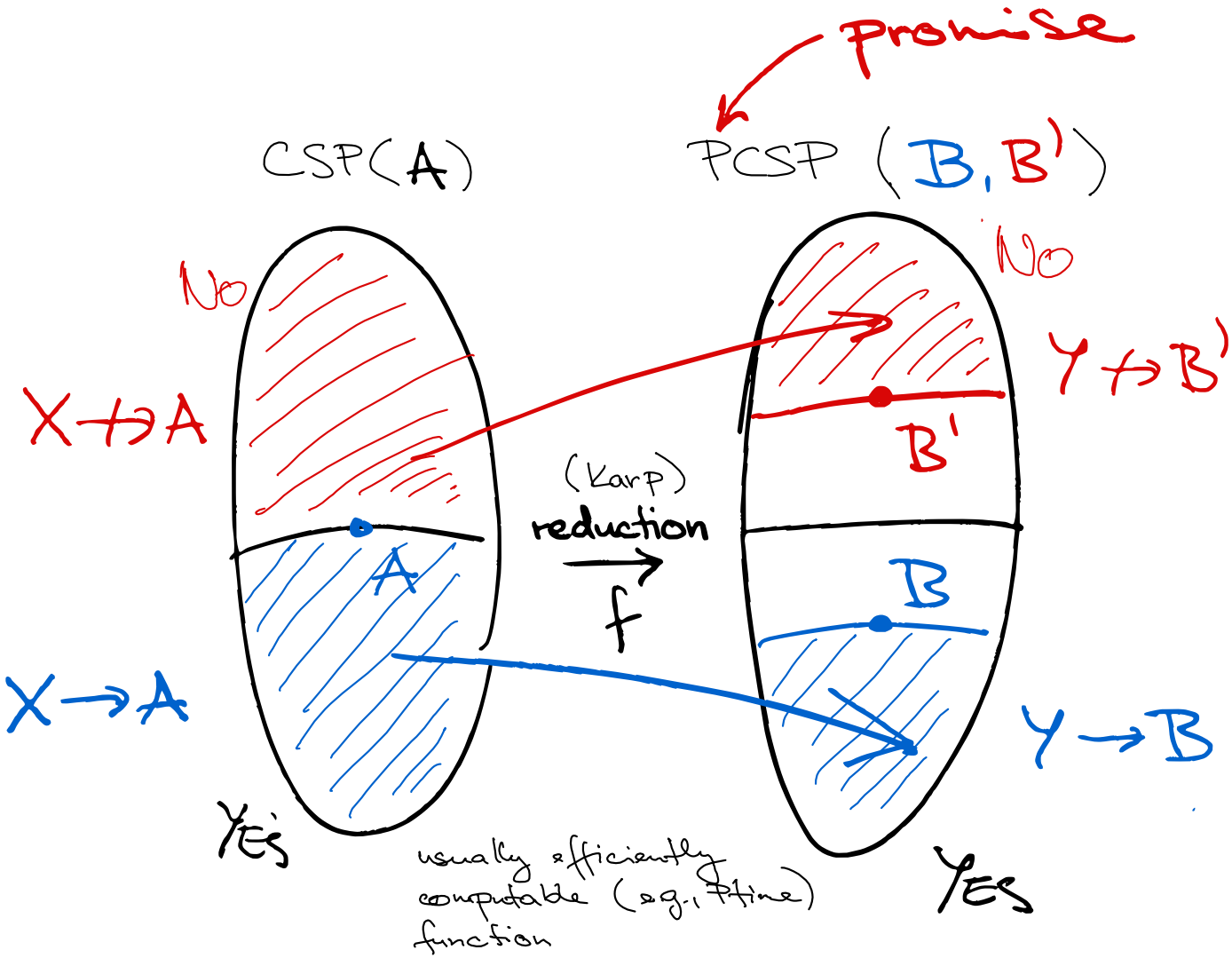
# Reductions & Promises




DEFINITION

A **promise** problem is to decide between disjoint (but not necessarily complementary) cases Yes / No

# Reductions & Promises

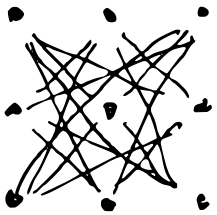


# Example. from 3-SAT to 3-colouring*

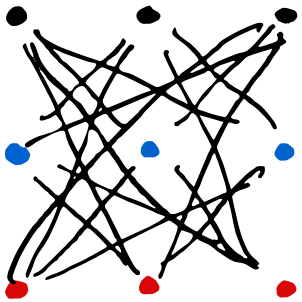
*a reduction due to
[Bulatov, Jeavons, Krokhin, 2005]

input:

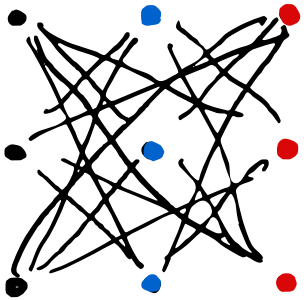
$(x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_4 \vee x_1) \wedge \dots$

$x_i \longmapsto$  $K_3^2$

$(x_1 \vee \neg x_2 \vee x_3) \longmapsto$  $K_3^7$



true



false

# Example. from 3-SAT to ~~3-colouring~~ *

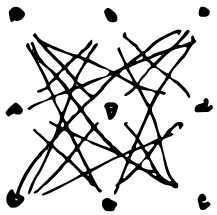
* a reduction due to [Bulatov, Jeavons, Krokhin, 2005]

PCSP $(K_3, K_4)$

[Brakensiek, Guruswami, 2016]
[Khanna, Linial, Safra, 2000]

input:

$(x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_4 \vee x_1) \wedge \dots$

$x_i \longmapsto$  $K_3^2$

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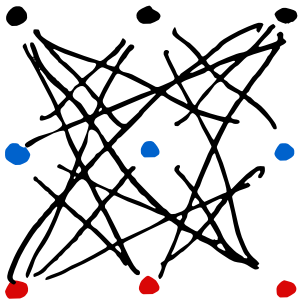

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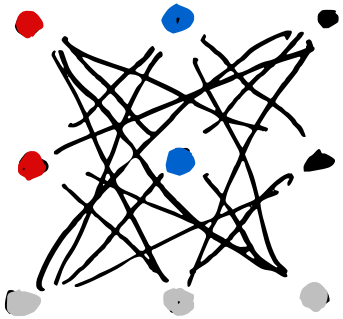
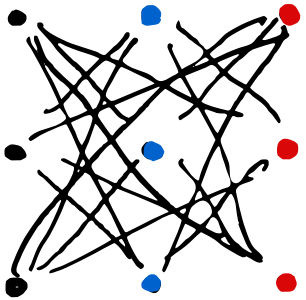
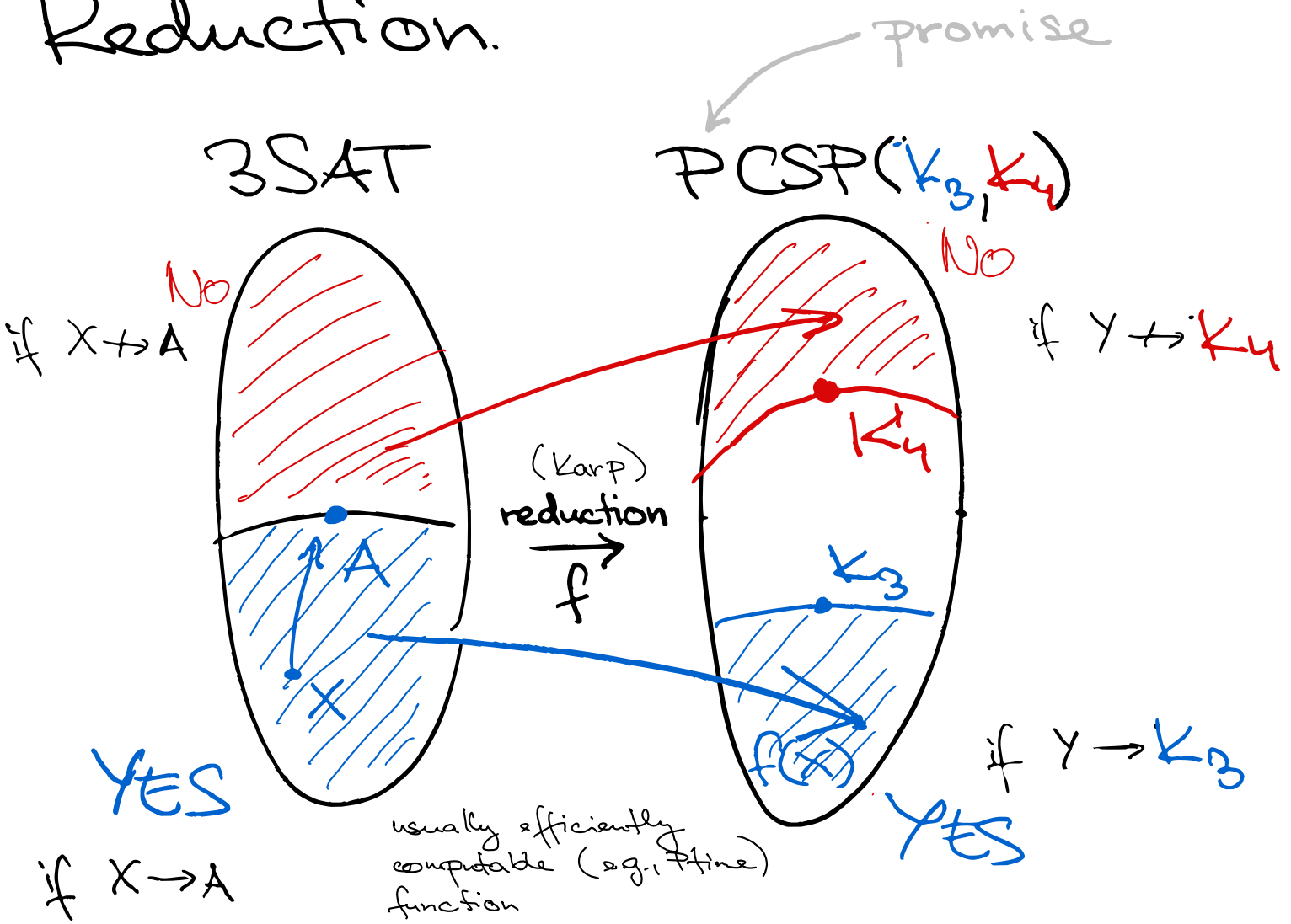




false

$3SAT \leq_{LogSpace} PCSP(K_3, K_4)$

# Reduction.




In fact <u>every CSP</u> is equivalent* to PCSP(P, M)

*unique up to gadget reductions.

Pol(A)

# Promise CSPs.

- Approximate graph colouring.
  given a graph $G$ that is $k$-colourable
  find a colouring with $c$ colours!
  (where $c > k$) $PCSP(K_k, K_c)$
  - $PCSP(K_3, K_5)$ is NP-hard
    [Bulin, Krokhin, O, 2019]
  - $PCSP(K_{2d}, K_{\binom{2d}{d}-1})$ is NP-hard
    [Wrochna, Živný, 2020]
  - $PCSP(K_3, K_6)$ is open!

- Promise 1-in-3SAT
  $1\text{-in-}3 = (\{0,1\}, \{(001),(010),(100)\})$
  Fix $B$ s.t. $1\text{-in-}3 \to B$, given a solvable 1-in-3SAT instance, find a homomorphism to $B$.
  - $PCSP(1\text{-in-}3, NAE) \in P$
  - $PCSP(1\text{-in-}3, \not\in) \in P$
    where $\not\in = (\not\in, x+y+z=1)$

  Conjecture (folklore)
  $PCSP(1\text{-in-}3, B)$ is NP-hard
  unless $\not\in \to B$

Always: $A \to B$

## Search $PCSP(A,B)$

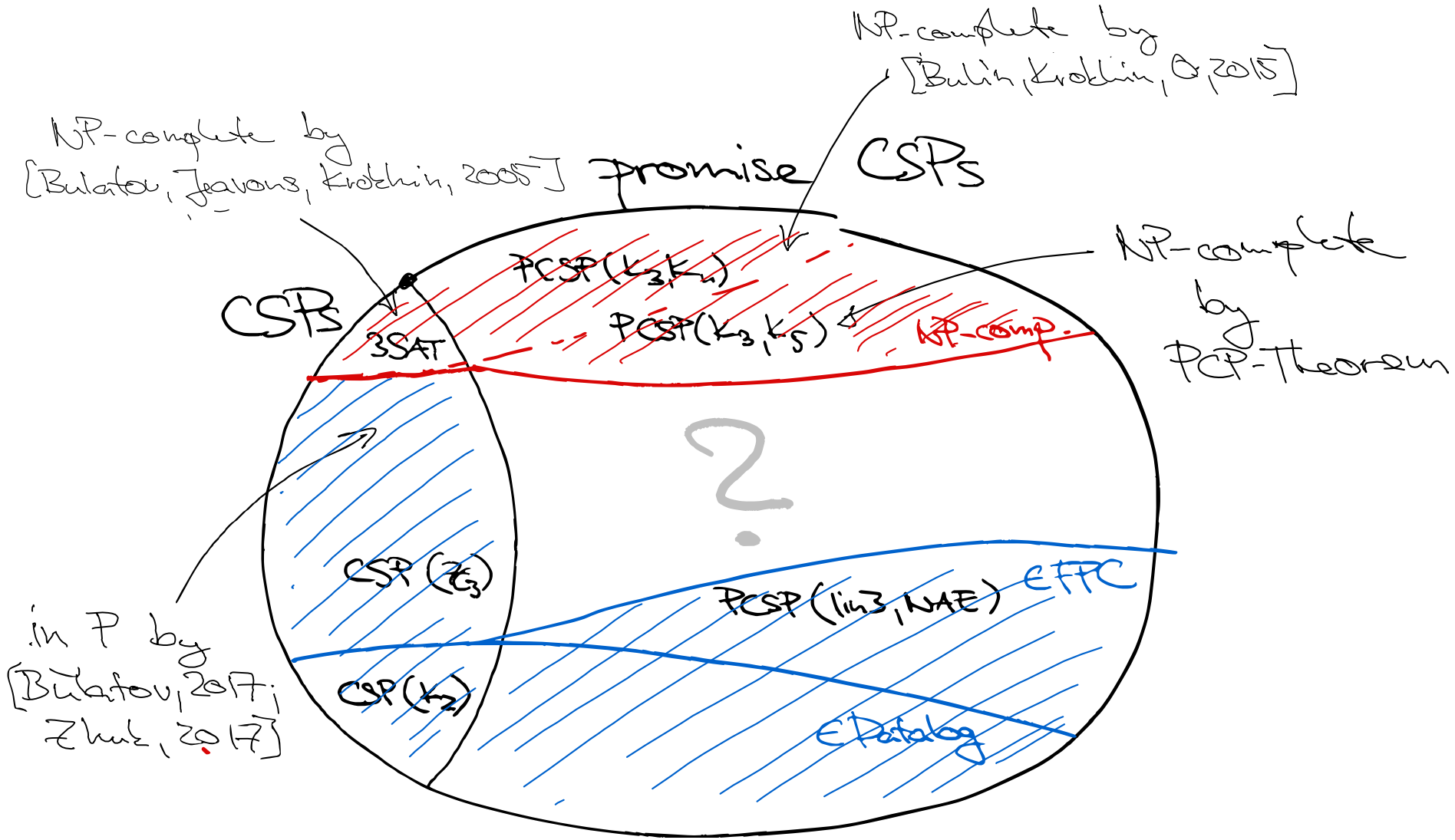
Given $X$ s.t. $X \to A$
find $h: X \to B$ ?

Open Problem.
Are these equivalent?

## Decision $PCSP(A,B)$

Given $X$ decide between:
YES: $X \to A$
NO: $X \not\to B$

# Overview


NP-complete by
[Bulatov, Jeavons, Krokhin, 2005]
NP-complete by
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promise CSPs
CSPs
3SAT
PCSP(K3,K4)
PCSP(K3,K5)
NP-comp.
NP-complete
by
PCP-theorem
?
in P by
[Bulatov, 2017;
Zhuk, 2017]
CSP(K3)
CSP(K2)
PCSP(lin3, NAE)
∈ FPC
∈ Datalog

# ALGORITHMS

## CSPs

- Bounded-width (Datalog) ✓
- Sheralı-Adams, etc ✓
- Mal'cev case [Bulatov; Dalmau, 2006] ✗
- Few subpowers [Idziak, et al., 2010] ✗
- Bulatov [2017] ✗
- Zhuk [2017] ✗

## promise CSPs

- open characterisation
- affine integer relaxation (AIP)
- combination of AIP+LP [Brakensiek, Guruswami, Wrochna, Živný, 2020]
- cohomological $k$-consistency [Adam]

AND MUCH MORE!

# New Directions

algebraic topology

[Krokhin, O, 2019]

+[Wrochna, Živný]

$PCSP(C_{1729}, K_3)$

graphs

↓

topological spaces (with a $\mathbb{Z}_2$-action)

↓

homotopy groups

promise CSPs

?

if $L(X) \to A$ iff $X \to R(A)$

(& $L$ is efficiently computable)

then

$PCSP(A, R(B)) \leq_L PCSP(L(A), B)$

$PCSP(K_{2d}, K_{\binom{2d}{d}-1})$

posetal adjunction

[Wrochna, Živný, 2020]

Everything that works for promise CSPs

works for non-promise CSPs

theorem (Bulatov-Zhuk)

CSP(A) is NP-complete iff Pol(A) → P

(assuming P ≠ NP)

THANK YOU!