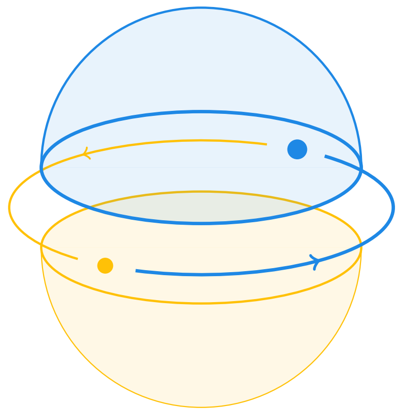
A topological approach to CSPs and its algebraic consequences

Jakub Opršal et al.



joint work with Sebastian Meyer (TU Dresden)

UNIVERSITY OF BIRMINGHAM

Why is algebraic topology so effective in computational complexity of CSPs?

Why is homotopy theory so effective in computational complexity of CSPs?

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for all $x, y \in A$.

Theorem [Taylor, 1977].

If an idempotent variety satisfies a non-trivial Maltsev condition, then it has a Taylor term.

$t\colon A^n \to A$ is *idempotent* if $t(x, \dots, x) \approx x$.

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Corollary [Siggers, 2010].
A locally finite variety with a Taylor term has a 6-ary term s satisfying

$$\begin{matrix} s(x & y & x & z & y & z) & \approx \\ s(y & x & z & x & z & y) & \end{matrix}$$

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Theorem [Hell, Nešetřil, 1990].
Unless P = NP, the only (non-trivial) H-colouring problem that is solvable in polynomial time is 2-colouring.

Algebraic proofs by:

- [Bulatov, 2005],
- [Siggers, 2010], and
- [Kun & Szegedi, 2016].

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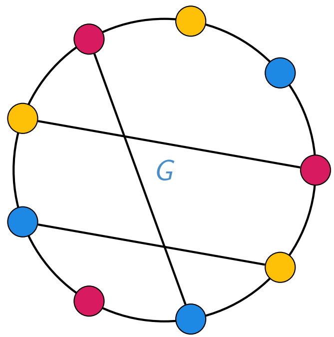
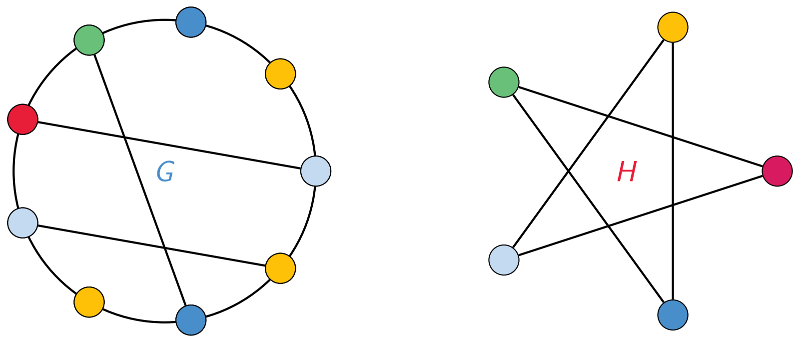
Why is homotopy theory so effective in computational complexity of CSPs?

Part I. What problems am I talking about?

Graph **colouring**



Graph colouring



Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, a graph homomorphism $G \to H$ is a mapping $h\colon V_G \to V_H$ that preserves edges,

$$uv \in E_G \Rightarrow h(u)h(v) \in E_H.$$

Example. A colouring of a graph G with k colours is just a homomorphism $c\colon G \to K_k$.

The H-colouring problem

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H-colouring

Fix a graph H (called *template*). Given a graph G, decide whether there is a homomorphism $G \to H$.

- K_2-colouring is easy (it is solvable in logspace [Reingold, 2005]);
- K_k-colouring is NP-complete for all $k > 2$.
- What about other graphs H?

Theorem [Hell & Nešetřil, 1990].
Unless P = NP, the only graph H-colouring problem that is solvable in polynomial time is 2-colouring.

Outline of a new proof

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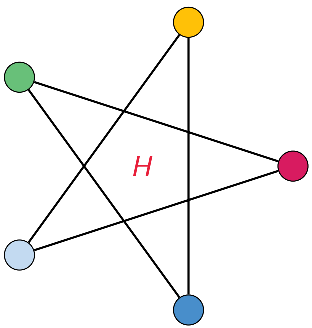


1. Identify which problems are NP-hard using the *algebraic approach to the constraint satisfaction problem*.
2. If *H*-colouring is not NP-hard, show that its *solution spaces* are component-wise contractible.
3. Use Brower's fixed-point theorem to show that *H* has a loop if *H* is not bipartite.

Why is homotopy theory so effective in computational complexity of CSPs?

Part II. What the . . . is the solution space of *H*-colouring?

Solution posets: Multihomomorphisms

A multihomomorphism is a function $f: V(G) \to 2^{V(H)} \setminus \{\emptyset\}$ such that, for all edges $uv \in E(G)$, we have that

$$f(u) \times f(v) \subseteq E(H).$$

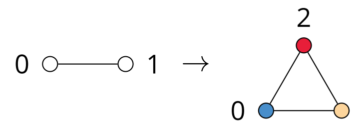
- Multihomomorphisms are naturally ordered

$$f \leq g \Leftrightarrow f(u) \subseteq g(u) \text{ for all } u$$

- $\mathrm{mhom}(G, H)$ is the poset of multihomomorphisms.

Solution posets: Multihomomorphisms

$K_2 \to K_3$



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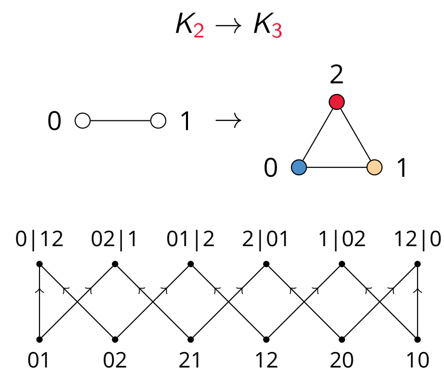

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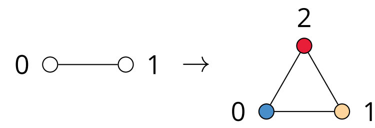
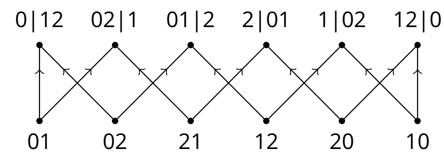
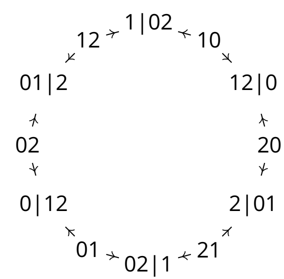






Solution spaces

Given graphs G and H, we define the space

$$\mathrm{Hom}(G, H) = |N\, \mathrm{mhom}(G, H)|$$

- The vertices are multihomomorphisms,
- f and g are connected by an arc if $f \leq g$,
- $\{f, g, h\}$ form a triangle if $f \leq g \leq h$,
- etc.

We view this as the solution space of instance G of H-colouring.

Example. $\mathrm{Hom}(K_2, K_3) \simeq S^1$.

Example. In $\mathrm{mhom}(K_2, K_4)$ we have:

$$01 \leq 02|1 \leq 02|13$$

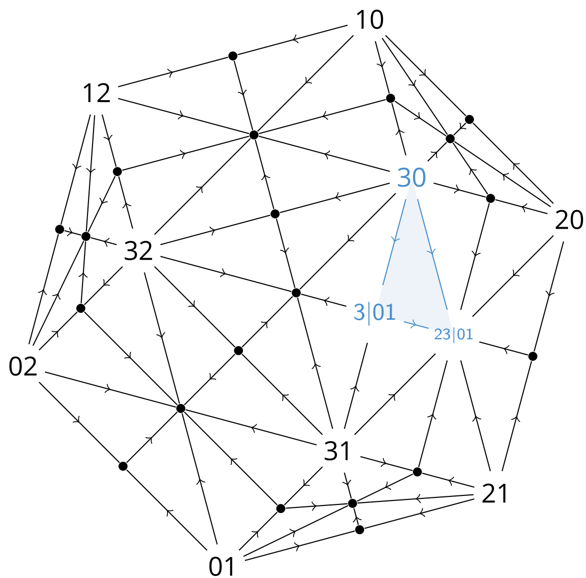
and

$$01 \leq 0|12 \leq 0|123$$

which creates 2-dimensional faces.

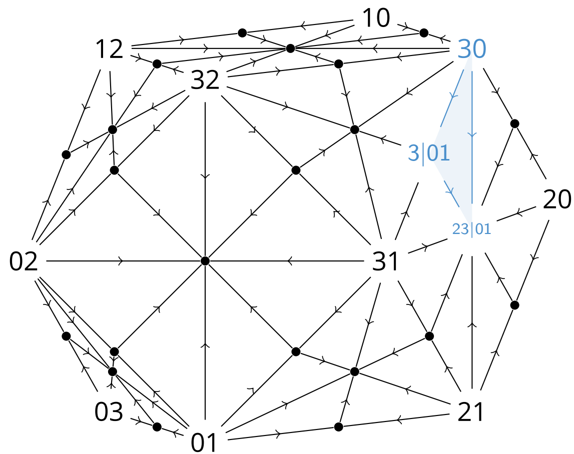
Two colourings f and g are connected if g can be obtained from f by **changing one value at a time** while remaining a valid colouring.

$\textcolor{red}{4}$-colourings of K_2



$\mathrm{Hom}(K_2, K_4)$

4-colourings of K_2



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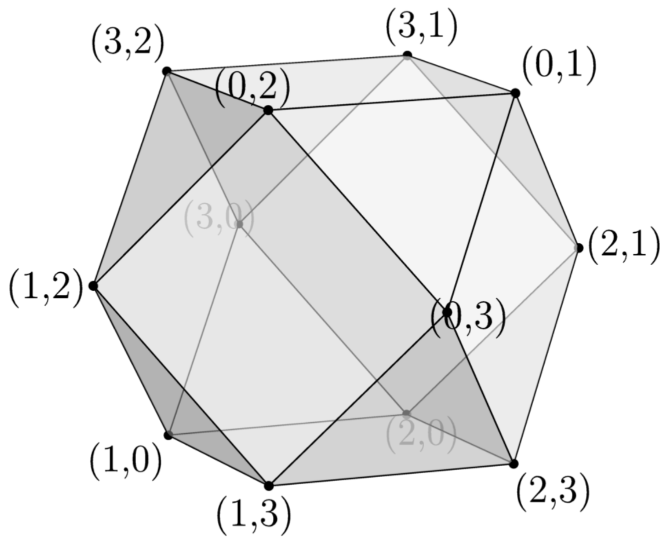

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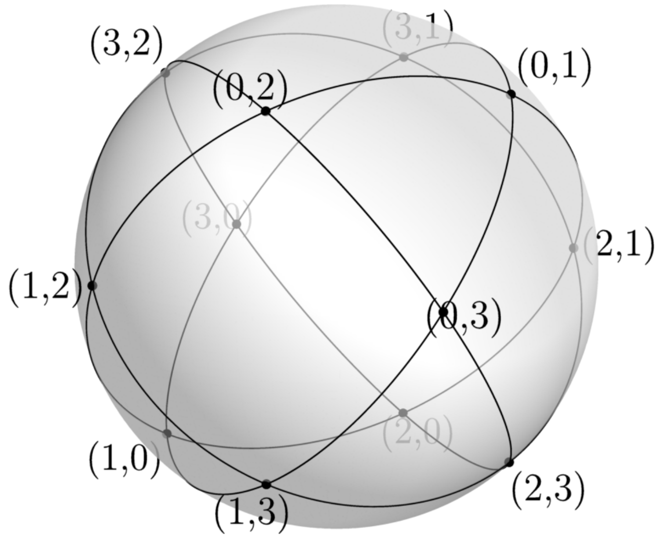

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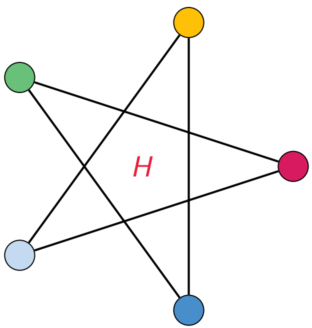
Why is homotopy theory so effective in computational complexity of CSPs?

Part III. A proof

Outline of the proof

Theorem [Hell, Nešetřil, 1990].
Unless P = NP, the only graph H-colouring problem that is solvable in polynomial time is 2-colouring.



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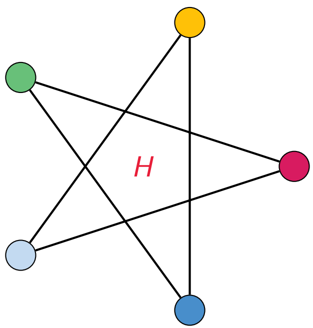

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Theorem (CSP Dichotomy).

A CSP with a finite template A either

1. *admits a Taylor polymorphism $t\colon A^n \to A$, and is in P [Bulatov, 2017; Zhuk, 2017]; or*
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Theorem [Bulatov, 2005; Siggers, 2005].

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Lemma [Taylor, 1977].

If a topological space X admits a continuous idempotent Taylor operation t, then $\pi_1(X)$ is Abelian.

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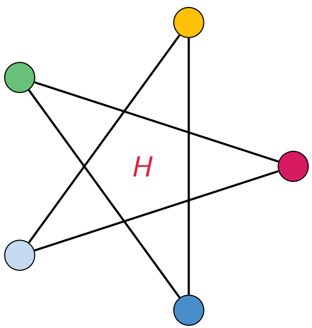

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Taylor $\rightarrow$ contractibility

A topological space X is called contractible if it is homotopy equivalent to a point $\{*\}$. For us, this is equivalent to $\pi_n(X) = 0$ for all $n \geq 0$.

Theorem [Larose, Zádori, 2005].
Every connected finite poset that admits a monotone Taylor operation is contractible.

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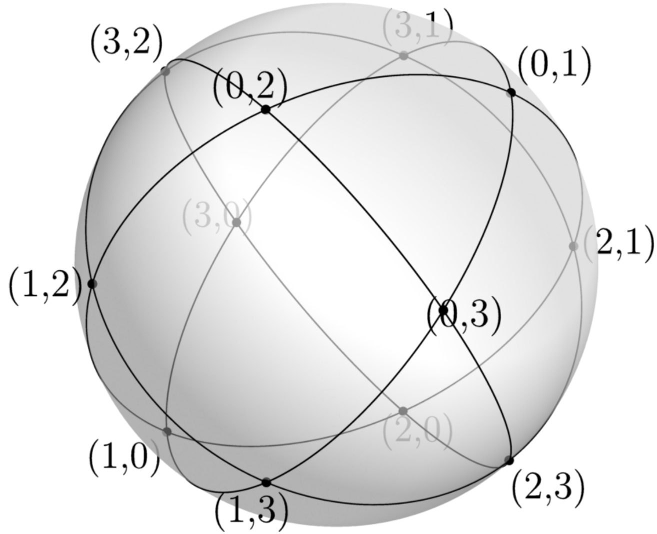
for all $x, y \in A$.

Theorem [Meyer, **O**, 2025].
Every connected finite poset that admits a monotone lax-Taylor operation is contractible, and therefore $\mathrm{Hom}(G, H)$ *is component-wise contractible for all* G *if* H *has a Taylor polymorphism.*

4-colourings of K_2



Hence, 4-colouring is NP-hard!

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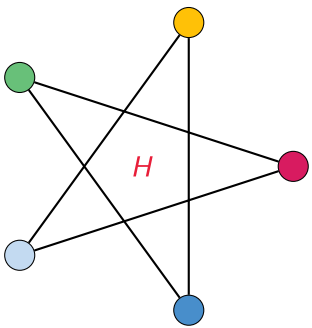

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A fixed-point theorem

Theorem (Brower's fixed-point theorem).

Every continuous function $f : D^n \to D^n$ *has a fixed point, i.e., there exists* $x \in D^n$ *such that* $f(x) = x$.

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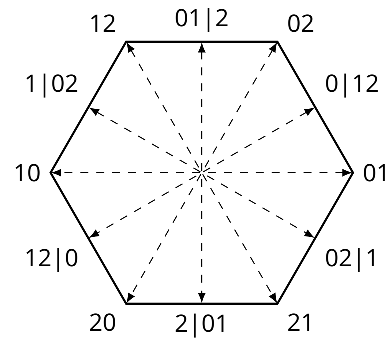
More generally: If X is a contractible compact CW-complex, then every function $f : X \to X$ has a fixed point.

A $\mathbb{Z}_2$ action on $\mathrm{Hom}(K_2, H)$

The space $\mathrm{Hom}(K_2, H)$ admits an action of the group $\mathbb{Z}_2$, i.e., there is a homeomorphism

$$\phi\colon\ \mathrm{Hom}(K_2, H) \to \mathrm{Hom}(K_2, H)$$

such that $\phi^2(x) = x$.



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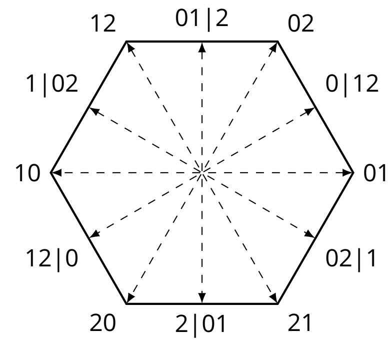

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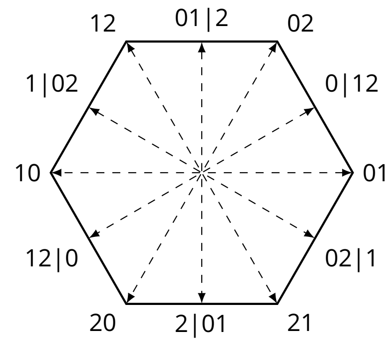

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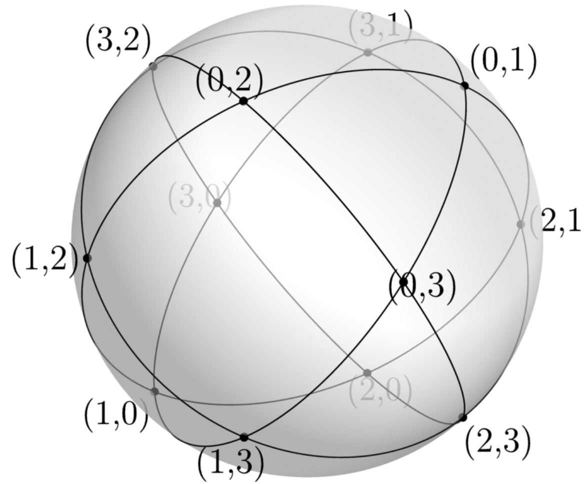

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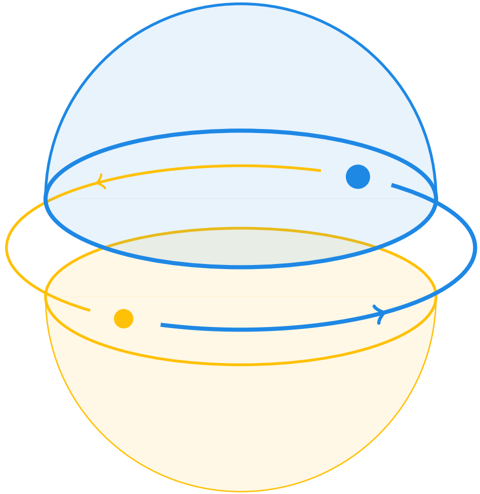

The proof

Theorem [Hell, Nešetřil, 1990].
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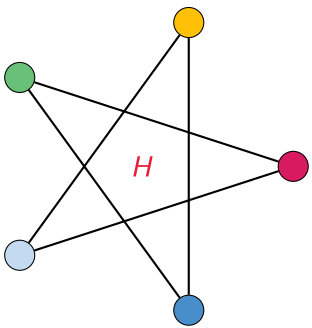

Proof. Assume that H is not-bipartite, and consider the space $X = \mathrm{Hom}(K_2, H)$.

Observe that the space admits a fixed-point free $\mathbb{Z}_2$-action $\phi\colon X \to X$ that for each multihomomorphism m flips the values of $m(0)$ and $m(1)$.

If H is not-bipartite then ϕ fixes a connected component of X. Indeed, if uv is an edge of an odd cycle of H then uv is connected to $vu = \phi(uv)$.

If H admitted a Taylor polymorphism, $\mathrm{mhom}(K_2, H)$ would admit a lax-Taylor operation, and all its connected component would be contractible.

Hence, ϕ which acts on the component of uv has a fixed point, the contradiction. ■

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- *a 12-ary term d satisfying*

$$\begin{aligned} d(y\ x\ x\ z\ x\ x\ x\ z\ y\ x\ y\ z) &\approx \\ d(x\ y\ x\ x\ z\ x\ y\ x\ z\ z\ x\ y) &\approx \\ d(x\ x\ y\ x\ x\ z\ z\ y\ x\ y\ z\ x) & \end{aligned}$$

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How does the topology of the solution space influence the complexity of a computational problem?

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Conjecture [Brakensiek, Guruswami, 2018].

Colouring graphs that are promised to map homomorphically to $C_{(2k+1)}$ with c colours, i.e., $\mathrm{CSP}(C_{(2k+1)}, K_c)$, is NP-complete for all $c > 2$ and $k > 0$.

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Theorem [Meyer, 2024].

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