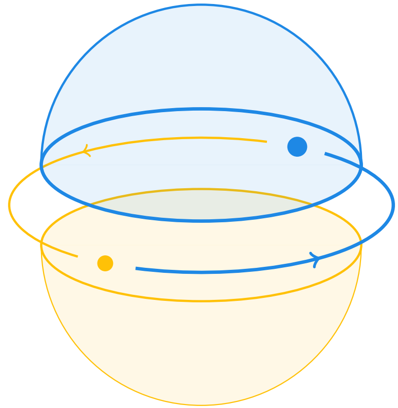
Homotopy and complexity of graph colouring

Jakub Opršal et al.



UNIVERSITY OF BIRMINGHAM

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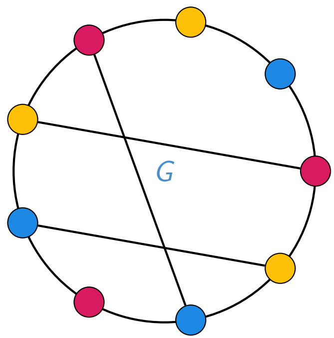
Part I. What problems am I talking about?

Graph colouring



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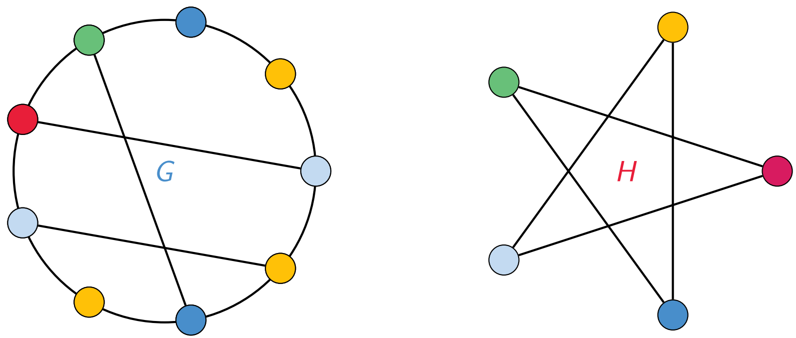

Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, a graph homomorphism $G \to H$ is a mapping $h\colon V_G \to V_H$ that preserves edges,

$$uv \in E_G \Rightarrow h(u)h(v) \in E_H.$$

Example. A colouring of a graph G with k colours is just a homomorphism $c\colon G \to K_k$.

The H-colouring problem

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H-colouring

Fix a graph H (called *template*). Given a graph G, decide whether there is a homomorphism $G \to H$.

- K_2-colouring is easy (it is solvable in logspace [Reingold, 2005]);
- K_k-colouring is NP-complete for all $k > 2$.
- What about other graphs H?

Theorem [Hell & Nešetřil, 1990].
Unless P = NP, the only graph H-colouring problem that is solvable in polynomial time is 2-colouring.

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Part II. What the … is the solution space of H-colouring?

Solution posets: Multihomomorphisms

A multihomomorphism is a function $f\colon V(G) \to 2^{V(H)} \setminus \{\emptyset\}$ such that, for all edges $uv \in E(G)$, we have that

$$f(u) \times f(v) \subseteq E(H).$$

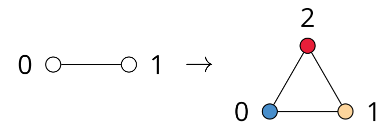
- Multihomomorphisms are naturally ordered

$$f \leq g \Leftrightarrow f(u) \subseteq g(u) \text{ for all } u$$

- $\mathrm{mhom}(G, H)$ is the poset of multihomomorphisms.

Solution posets: Multihomomorphisms

$K_2 \to K_3$



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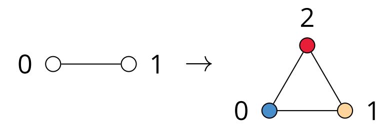

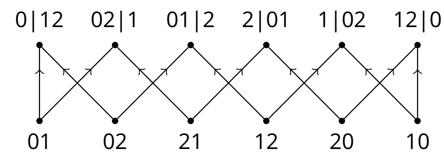

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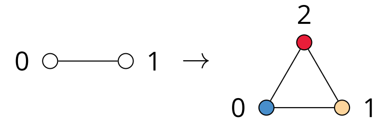
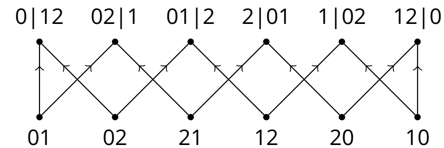
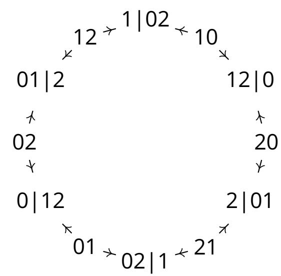






Solution spaces

Given graphs G and H, we define the space

$$\mathrm{Hom}(G, H) = |N\,\mathrm{mhom}(G, H)|$$

- The vertices are multihomomorphisms,
- f and g are connected by an arc if $f \le g$,
- $\{f, g, h\}$ form a triangle if $f \le g \le h$,
- etc.

We view this as the solution space of instance G of H-colouring.

Example. $\mathrm{Hom}(K_2, K_3) \simeq S^1$.

Example. In $\mathrm{mhom}(K_2, K_4)$ we have:

$$01 \le 02|1 \le 02|13$$

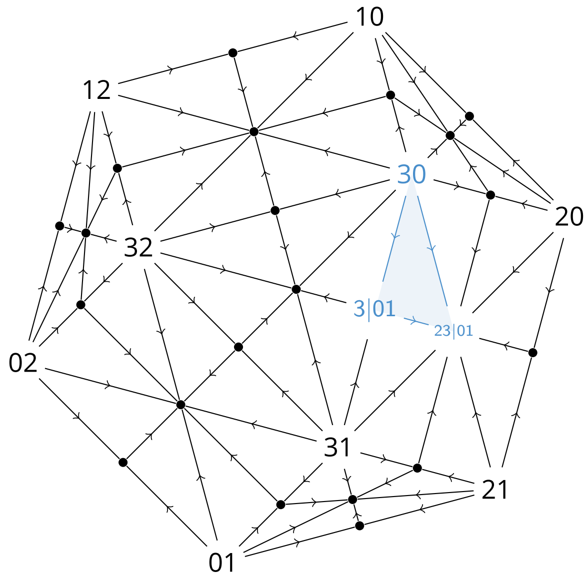
and

$$01 \le 0|12 \le 0|123$$

which creates 2-dimensional faces.

Two colourings f and g are connected if g can be obtained from f by **changing one value at a time** while remaining a valid colouring.

4-colourings of K_2



$\mathrm{Hom}(K_2, K_4)$

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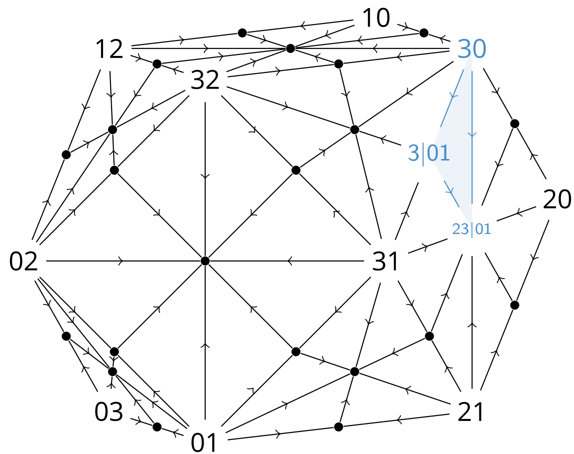

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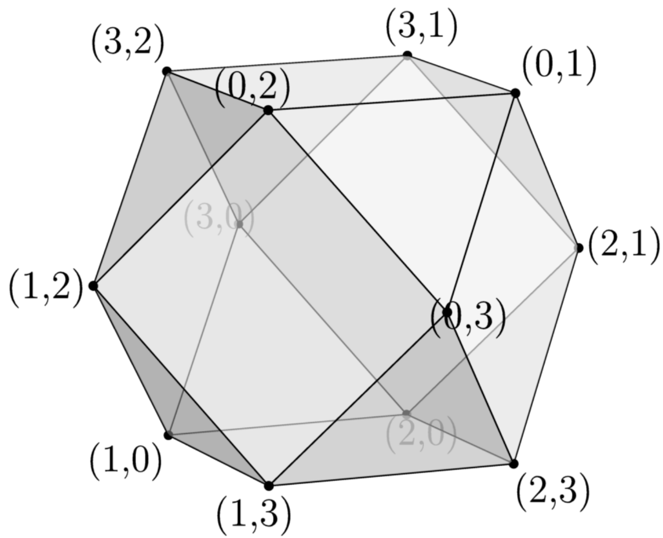

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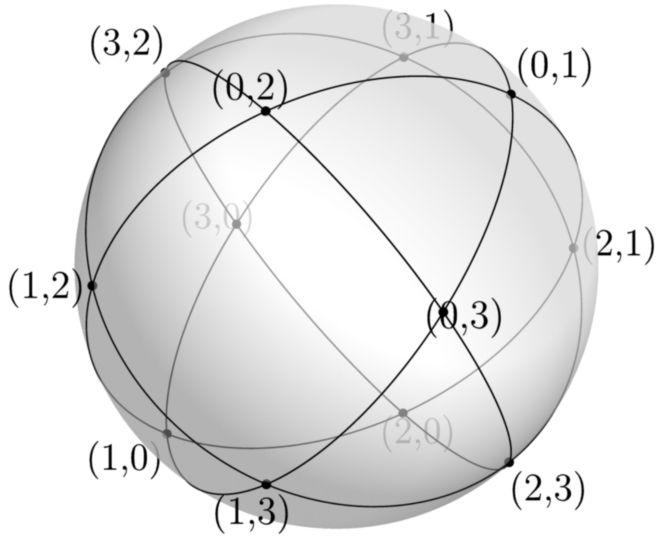

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Part III. What can we say about complexity in terms of topology?

Topological methods in complexity of graph colouring

Conjecture [Brakensiek, Guruswami, 2018].

Colouring graphs that are promised to map homomorphically to $C_{(2k+1)}$ with c colours is NP-complete for all $c > 2$.

[1] Krokhin, **O** (2019). The complexity of 3-colouring H-colourable graphs. *Symposium on Foundations of Computer Science, FOCS 2019*.

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Theorem.

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Theorem [Hell, Nešetřil, 1990].
Unless P = NP, the only graph H-colouring problem that is solvable in polynomial time is 2-colouring.

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Part IV. A proof

Outline of the proof

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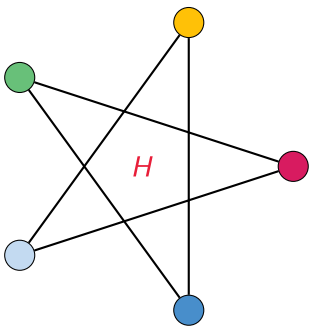

1. Identify which problems are NP-hard using the *algebraic approach to the constraint satisfaction problem*.
2. If *H*-colouring is not NP-hard, show that its solution spaces are component-wise contractible.
3. Use Brower's fixed-point theorem to show that *H* has a loop if *H* is not bipartite.

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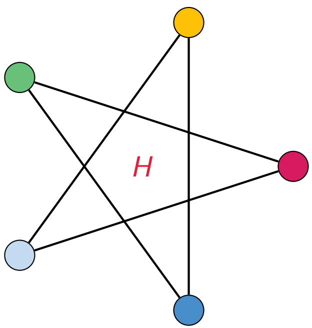

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Algebraic approach to the constraint satisfaction problem

An operation $t\colon A^n \to A$ is Taylor

$$
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t(x \quad * \quad \dots \quad *) &= t(y \quad * \quad \dots \quad *)\\
t(* \quad x \quad \dots \quad *) &= t(* \quad y \quad \dots \quad *)\\
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Theorem (CSP Dichotomy).

A CSP with a finite template A is either

1. *admits a Taylor homomorphism $t\colon A^n \to A$, and is in P [Bulatov, 2017; Zhuk, 2017]; or*
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Theorem [Bulatov, 2005; Siggers, 2005].

A (core) graph H has a Taylor homomorphism if and only if it is bipartite.

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for all $x, y \in A$.

Lemma [Taylor, 1977].

If a topological space X admits a continuous idempotent Taylor operation t, then $\pi_1(X)$ is Abelian.

$t\colon A^n \to A$ is *idempotent* if $t(x, \dots, x) = x$.

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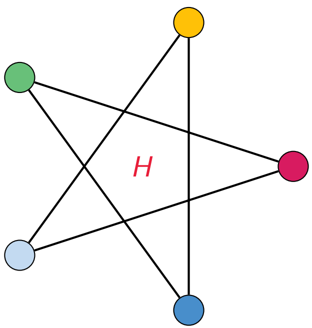

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Taylor $\to$ contractibility

A topological space X is called contractible if it is homotopy equivalent to a point $\{*\}$.
For us, this is equivalent to $\pi_n(X) = 0$ for all $n \geq 0$.

Theorem [Larose, Zádori, 2005].
Every connected finite poset that admits a monotone Taylor operation is contractible.

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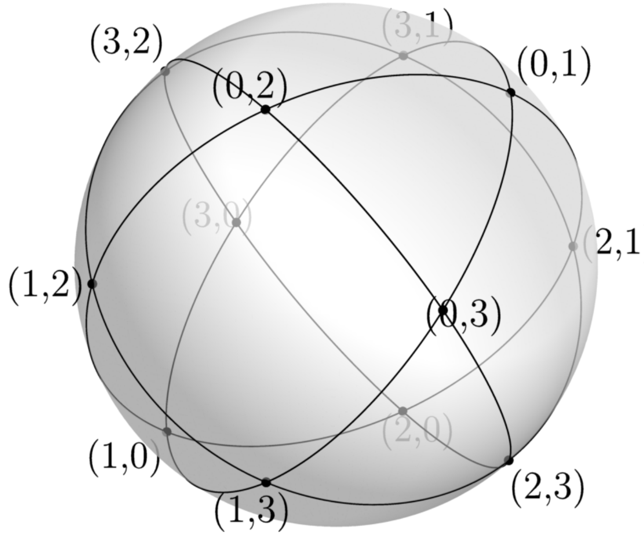
Theorem [Meyer, 2024; Meyer, **O**, 2025].
If H admits a Taylor homomorphism, then $\mathrm{Hom}(G, H)$ is component-wise contractible for all G.

Therefore, unless H-colouring is NP-hard, all **solution spaces are component-wise contractible!**

4-colourings of K_2



Hence, 4-colouring is NP-hard!

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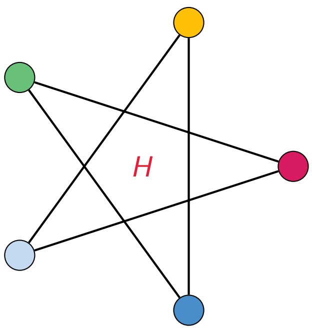

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A fixed-point theorem

Theorem (Brower's fixed-point theorem).

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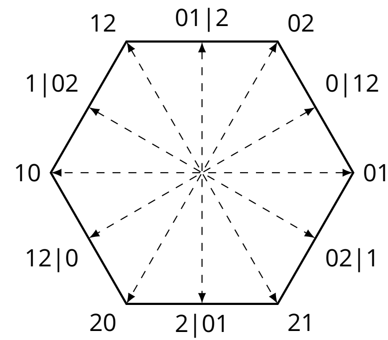
More generally: If X is a contractible compact CW-complex, then every function $f : X \to X$ has a fixed point.

A $\mathbb{Z}_2$ action on $\mathrm{Hom}(K_2, H)$

The space $\mathrm{Hom}(K_2, H)$ admits an action of the group $\mathbb{Z}_2$, i.e., there is a homeomorphism

$$\phi\colon\ \mathrm{Hom}(K_2, H) \to \mathrm{Hom}(K_2, H)$$

such that $\phi^2(x) = x$.



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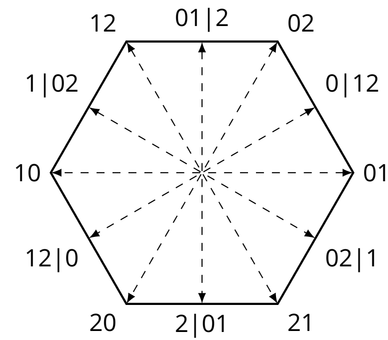
- flipping the two values induces a monotone involution on the poset, and hence a continuous involution on the space.



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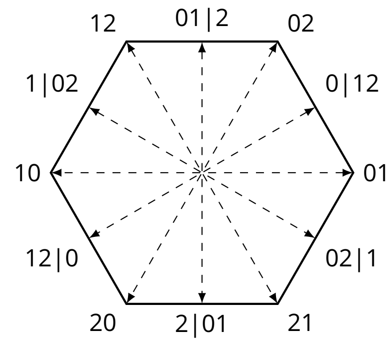
- ϕ does not have a fixed point, otherwise H would contain an edge uu.



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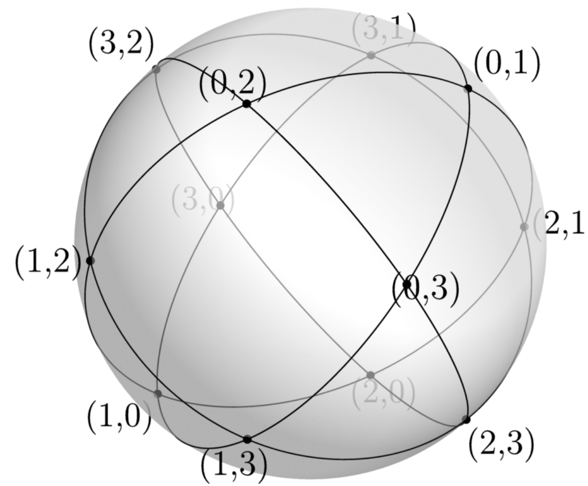

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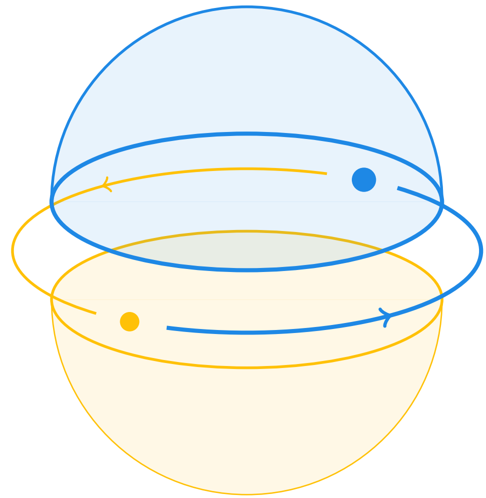

The proof

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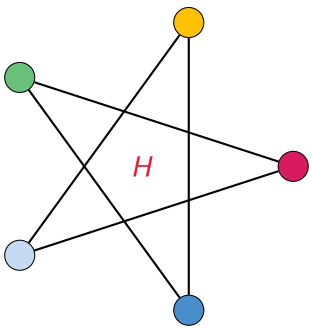

Proof. Assume that H is not-bipartite, and consider the space $X = \mathrm{Hom}(K_2, H)$.

Observe that the space admits a fixed-point free $\mathbb{Z}_2$-action $\phi\colon X \to X$ that for each multihomomorphism m flips the values of $m(0)$ and $m(1)$.

If H is not-bipartite then ϕ fixes a connected component of X. Indeed, if uv is an edge of an odd cycle of H then uv is connected to $vu = \phi(uv)$.

If H admitted a Taylor homomorphism, $\mathrm{mhom}(K_2, H)$ would admit a lax-Taylor operation, and all its connected component would be contractible.

Hence, ϕ which acts on the component of uv has a fixed point, the contradiction. ∎

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Theorem [Meyer, 2024; Meyer, **O**, 2025].

A constraint satisfaction problem is NP-complete, unless each connected component of the solution space is contractible (i.e., topologically trivial).

Corollary [Hell, Nešetřil, 1990].

Unless P = NP, the only graph H-colouring problem that is solvable in polynomial time is 2-colouring.

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[1] Krokhin, **O** (2019). The complexity of 3-colouring H-colourable graphs. *Symposium on Foundations of Computer Science, FOCS 2019*.

[2] Wrochna, Živný (2020). Improved hardness for H-colourings of G-colourable graphs. *Symposium on Discrete Algorithms, SODA 2020*.

[3] Avvakumov, Filakovský, **O**, Tasinato, & Wagner (2025). Hardness of 4-colouring G-colourable graphs. *Accepted to STOC 2025*.

Theorem.
Colouring graphs that are promised to map homomorphically to $C_{(2k+1)}$ with 4 colours is NP-complete.

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[5] Meyer, **O** (2025). A topological proof of the Hell–Nešetřil dichotomy. *Symposium on Discrete Algorithms, SODA 2025*.

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